INSTANTANEOUS LIABILITY RULE AUCTIONS: THE CONTINUOUS EXTENSION OF HIGHER-ORDER LIABILITY RULES

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ABSTRACT

A "higher-order" liability regime—in which a plaintiff and a defendant have a sequence of alternating options to take (or to put) a disputed entitlement —can enhance allocative efficiency by harnessing the private information possessed by both litigants. Indeed, infinite order liability regimes can, as a theoretical matter, assure first-best efficiency. Such iterated taking regimes have, however, been criticized as (i) generating excessive (and debilitating) taking costs, and (ii) being infeasible with regard to intangible entitlements. This Paper shows that courts can replicate the first-best efficiency of infinite-stage liability rule via an instantaneous auction mechanism. This instantaneous mechanism avoids the excessive taking cost criticism (because the disputants merely submit a single report of value). Unlike many auctions, the mechanism also allows courts to pursue equitable goals by dividing the bulk of expected gains to either plaintiff or defendant (without undermining the first-best allocative efficiency). A derivation is given of the explicit formula for the basic element of the procedure, which we call the "damage curve" and which determines the amount of damages that the winner of the auction must pay the loser. This formula holds for arbitrary joint probability distributions of the valuations of the asset, whether correlated or uncorrelated. Explicit damage curves are calculated for several concrete examples, illustrating both correlated and uncorrelated cases.

I. Introduction

Kaplow and Shavell[4]—formalizing Calabresi and Melamed[5]—showed that "liability rules" can harness the private information of a potential taker to enhance allocative efficiency. For example, when

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[4]Louis Kaplow and Steven Shavell, *Property Rules Versus Liability Rules: An Economic Analysis,* 109 Harv. L. Rev. 713 (1996).

[5]Guido Calabresi and Douglas Melamed, *Property Rules, Liability Rules, and Inalienability: One View of the Cathedral,* 85 Harv. L. Rev. 1089 (1972).

nuisance damages are set at the expected value of the pollutee (takee) a potential polluter (taker) will be induced to take only if she expects that the taking would enhance efficiency.

But while simple liability rules can do a better job at economizing on the private information of potential takers than property rules can, Ayres and Balkin[6] pointed out that they fail to harness the private information of the other side of the dispute. Ayres and Goldbart[7] showed that giving the disputants a sequence of alternating options to take a disputed entitlement at successively increasing court ordered damages could even better enhance allocative efficiency by harnessing the private information of both parties. These "higher order" liability rules resembled an auction in which each successive taking amounted to a bid signaling a higher private value. And indeed, a potentially infinite sequence of takings could, in theory, just like a traditional auction, produce first-best allocative efficiency—with the disputed entitlement always being allocated to the higher valuer.

But, unlike traditional auctions, where the winning bidder pays a third party (the seller), this regime represented an internal auction, in which the winning bidder paid the losing bidder. Ayres and Goldbart[8] showed how this internal auction feature enhanced the distributional flexibility of courts to respect the equitable claims of the polutee or the polluter (or enhance *ex ante* investment incentives). Indeed, it is possible to construct a higher-order liability rule so as to maintain first-best allocative efficiency and divide the expected value of the entitlement between the disputants as the court sees fit.

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[6]I. Ayres and J.M. Balkin, *Legal Entitlements as Auctions: Property Rules, Liability Rules, and Beyond, 106* Yale L.J. *703, 729-33 (1996).*

[7]I. Ayres and P.M. Goldbart, *Optimal Delegation and Decoupling in the Design of Liability Rules, 100* Mich. L. Rev., *1-79 (2001).*

[8]See *supra* note *7.*

While infinite staged higher-order liabilityrules thus have attractive theoretical properties, they have been criticized as being impractical. Kaplow and Shavell[9] point out that iterated taking regimes might (i) generate excessive (and debilitating) taking costs and (ii) be infeasible with regard to intangible entitlements.[10]

The present article advances the ball by showing that it is possible to implement an infinitestage liabilityrule with an instantaneous procedure – where litigants make a single report of their valuation to the court – that avoids the takings problems identified by Kaplow and Shavell. Our procedure achieves first-best efficiency in a model without any possibility of consensual trade. Ours is a direct mechanism in which the disputants are asked to report how much they value the entitlement to the court, with knowledge that the court will (i) allocate the entitlement to the disputant submitting the higher report and (ii) assess damages according to a pre-specified damage curve (that is a function of both disputant's reports).

We show that there exists an equilibrium in which disputants report their true values and the entitlement is accordingly allocated by the court to the first-best valuer. We present a derivation of the explicit formula for the core feature of the procedure, what we call the "damage curve", which determines the amount of damages that the winner of the auction must pay the loser. This formula holds for arbitrary joint probability distributions of the valuations of the asset, whether correlated or uncorrelated. We calculate explicit damage curves for several concrete (correlated and uncorrelated) examples.

There are of course other auction mechanisms that could also achieve first-best allocative efficiency.

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[9]See *supra* note 4.

[10]These criticism are refuted in Ian Ayres, Optional Law: Real Options in the Structure of Legal Entitlements (forthcoming University of Chicago Press, 2005).

For example, the government could (i) exercise its eminent domain option over the disputed entitlement (paying the *ex ante* expected value of the entitlement to one of the disputants), and then (ii) auction the entitlement to the highest bidder (with the government keeping the revenues of the auction). But unlike this two step process (where the government first takes then auctions), our paper shows that it is possible to harness the private information of the litigants in a one-shot internal auction (in which the winner compensates the loser). Moreover, the government knows more about the value of the entitlement at the end of the auction than at the beginning, because the very process of bidding reveals information about the disputants' private valuations. Accordingly, an advantage of our proposal over other auction mechanisms is that it allows more nuanced divisions of the entitlement's ultimate value between the disputants.[11]

A. Review of higher-order liability rules

Once scholars noticed that traditionally liability rules implicitly grant a call option to one of the litigants, it become natural to ask whether this option entitlement should itself be protected by a property or a liability rule. After one party exercises its option to take nonconsensually, should the other have an option to "take back"? Almost all analyses of liability rules have implicitly assumed that the law deters the initial entitlement holder from taking back after an initial nonconsensual taking. For example, if a liability rule regime gives Calabresi an option to take some entitlement of Melamed for $100, most analysts assume that after this taking, Melamed (and others) would not have a viable option to take the entitlement back from Calabresi. In other words, most people have assumed that liability rules are protected by property rules.

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[11]However, a still unresolved question is whether these higher-order mechanisms dominate a host of other mechanisms that have recently been proposed to enhance efficiency. *See, e.g.,* Richard R.W. Brooks, Simple Rules for Simple Courts (working paper 2003).

Kaplow and Shavell were right to see that liability rules harness the *taker's* private information. But traditional liability rules do nothing to harness the private information of the *takee*. Giving the original entitlement holder a take-back option can result in second-order takings that produce even greater efficiency, because they better economize on both parties' private information. Protecting a liability rule option with a liability rule can be more efficient than the traditional (single-chooser) liability rule without such a take-back option.

As a matter of nomenclature, we will refer to the traditional (single-chooser) liability rule of Calabresi and Melamed as a "first-order" rule because it contemplates at most one nonconsensual taking. And by analogy, we will call a regime where the entitlement holder has a take-back option a "second-order" liability rule, because this rule presumes the possibility of two nonconsensual takings. Under a second-order liability regime, a potential polluter would have an option to pay the original entitlement owner a predetermined sum for the right to pollute. However, before pollution began, the original owner would then have the option to pay the polluter an even larger sum to maintain the status quo ante. In a second-order liability regime, once the original owner had exercised her take-back option, property rule protection would henceforth deter the polluter from polluting.

Of course, we need not stop with second-order takings. It is theoretically possible to consider third- or higher-order liability rules involving a longer series of reciprocal taking options. Higher-order liability rules (with multiple taking options) can implement an efficient *auction*–where each taking represents a "bid" signaling a higher valuation. Auctions can be structured with a variety of rules,[12] but for present

[12]Auctions can be implemented with either sealed bidding or open-call bidding, and open-call bidding can be accomplished with either ascending bids (as in so-called "English" auctions) or with descending bids (as in so-called "Dutch" auctions). In "second-price auctions," winning bidders sometimes must only pay the second-highest

purposes it will be particularly useful to focus on two aspects of auction design: the size of the minimum (ascending) bid increments, and the rules for distribution of proceeds.[13]

1. Internal auctions and examples of higher-order regimes

In the most familiar auction situation, winning bidders pay a third party (i.e., the seller), and not each other, but this is not a necessary rule for distributing the proceeds of an auction.[14] But reciprocal taking option regimes, where the winning "bidder" pays the losing "bidder," can produce the same allocational result as a traditional auction with minimum bid increments. Higher-order liability rules represent a kind of "internal" auction in which the auction proceeds are distributed internally among the auction bidders.[15] An arbitrarily larger number of reciprocal taking options will produce an internal auction with an arbitrarily small bid increment--which in the limiting case produces first-best efficiency.

This auction reinterpretation reveals that liability and property rules are also special cases of a larger family of truncated auctions. Traditional (first-order) liability rules are one-round auctions where we expect at most one bid. We can even think of property rules as zero-round auctions, because the law deliberately sets the initial exercise price above the highest valuation expected of all potential takers. A property rule

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bid, instead of what they bid themselves. *See id*. at 230(describing auction variants).

[13]One might think that another important consideration would be the number of possible rounds. In the examples we consider here, however, the parties' maximum valuation of the entitlement is already known. Hence, the number of possible rounds is largely dictated by the size of the bidding increments.

[14]For example, in a popular class exercise, a professor offers to auction a $10 bill to the highest bidder—with the important catch that both the first-and second-highest bidders are required to pay. Once the bidding hits $10, the second-highest bidder suddenly realizes that it is better to bid $11 to win the auction (and thereby lose $1) than to come in second and lost $9. For a real world example of this "war of attrition" auction, *see* Ian Ayres & Peter Cramton, *Pursing Deficit Reduction Through Diversity: How Aflitigantative Action at the FCC Increased Competition*, 48 STAN. L. REV. 761 (1996).

[15]We distinguish this from the more familiar situation of an "external auction," where the parties bid for an entitlement owned by another and the winner pays the owner for it.

is an auction in which the minimum initial bid is simply set too high.[16]

The more rounds we add to an internal auction, the more it appears to mimic Coasean bargaining between the participants. Although the allocative efficiency of the internal auction produced by higher-order liability rules resembles the allocative efficiency of Coasean bargaining, the mechanisms differ in three important ways.

First, internal auctions can be more efficient precisely because bargaining between individuals is not always practical. Lack of information and other transaction costs may prevent efficient bargains from being struck. The great advantage of auctions over unstructured bargaining lies in the way that they set clear choices and structure responses. In this fashion they compensate for the imperfections that block efficient negotiation.[17] Higher-order liability rules can force the parties to reveal information about their valuations and help produce results closer in efficiency to those that might have been achieved through bargaining with full information and under ideal conditions.

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[16]Or, to put it another way, a property rule is like an auction at Sotheby's where the owner really does not want to part with the painting, and thus requires an exceptionally high opening bid. In real life, the auction house will advise (or require) that the initial bid be set lower, because it wants to move merchandise and collect a percentage of the bid. But in this respect the legal system differs from the owner of an auction house; it may have good reasons to respect the desire of the owner not to surrender the chattel except consensually and at the owner's asking price. *See infra* text accompanying notes 36-38; Part VI.

[17]The possibility of inefficient bargaining is dramatized by what economists call *bilateral monopoly*:

> Bilateral monopolies, which arise when two parties are locked into dealing with each other. . . can give rise to high negotiation costs that foreclose efficient transfers. Because there is no competitive pressure from outsiders, each party is likely to bargain "strategically"—asking much, offering little, bluffing, threatening to walk away from the deal—in an effort to get as much as possible. . . . "[B]ilateral monopoly is a social problem, because the transaction costs incurred by each party in an effort to engross as much of the profit of the transaction as possible are a social waste. They alter the relative wealth of the parties but do to increase the aggregate wealth of society. A major thrust of the common law . . . is to mitigate bilateral-monopoly problems."

Jessie Dukeminier & James E. Krier, PROPERTY 137 n. 17 (3d ed. 1993)(quoting Richard A. Posner, ECONOMIC ANALYSIS OF LAW 62 (4th ed. 1992)). Higher-order liability rules may be able to mitigate bilateral monopoly problems in settings that otherwise seem to have low transaction costs.

Second, bargaining between individuals in a property rule regime is consensual, but internal auctions are not. In face-to-face bargaining, the parties do not have to transfer their entitlements unless they agree to do so. However, under a higher-order liability regime, the entitlement holder might have her entitlement taken at any time without her consent. The taker, in turn, can have the entitlement retaken without her consent, and so on. In a truly consensual arrangement, parties can simply refuse to deal if they do not want to part with their existing entitlements. However, once an internal auction is set in motion by a party's nonconsensual taking, the takee may not be able to bargain her way out of the process. She may not be able to keep her entitlement unless she retakes.[18] Thus, higher-order liability rules can produce greater efficiency precisely in those cases where Coasean bargaining under ideal conditions is impractical.

Third, the internal (revenue sharing) aspect of this auction mechanism leads litigants to be less guarded in revealing their true valuation. William Samuelson has proven that there is no bargaining (or non-bargaining) mechanism that will induce allocative efficiency when the disputants have private information about their values and the entitlement protected by a property rule is assigned exclusively to one side.[19] But higher-order liability rules because of this internal revenue sharing divide the claims to the entitlement between the two litigants and thus offer the potential of producing first-best allocative efficiency.[20]

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[18]In this respect, an internal auction differs from the familiar "highest bidder" auction that culminates in a consensual trade between the highest bidder and a third party. In these traditional auctions, participation is consensual in the sense that one does not have to bid; only those who participate and win pay proceeds to a third party, producing a result similar to a bargain freely entered into between them. But this case forms only a small class of possible regimes. For example, in third party auctions where the penultimate bidder must also pay, the parties may not be able to walk away so easily once the bidding starts.

[19]William Samuelson, *Bargaining Under Asymmetric Information*, 52 ECONOMETRICA 995 (1984).

[20]Chapter 2 showed how first-order options divided the entitlement between the litigants. *See* supra *at* 23. *See* Peter Cramton et al., *Dissolving a Partnership Efficiently*, 55 ECONOMETRICA 615 (1987) (showing how divided claims to a partnership might lead to first best bargaining) and Ayres & Talley I, *supra* note 30 (showing how divided claims to entitlements more generally might enhance allocative efficiency in bargaining).

Notions of higher-order liability rules with reciprocal taking options will strike many readers as strange and unworldly. To give these abstract notions a slightly more human face (and especially before proceeding to introduce some rather intimidating formulas), we pause briefly to provide two examples of second-order liability rules. The first is an existing common law rule, and the second is a proposal for a modification of a common law rule made several years back by one of the titans of property law, Robert Ellickson.

A good example of a second-order liability rule in the common law is the incomplete privilege of private necessity available in cases of intentional tort.[21] In the famous case of *Vincent v. Lake Erie Transportation Co.*,[22] the Minnesota Supreme Court used the doctrine of incomplete privilege to hold a shipowner liable when his ship damaged a dock while he attempted to moor the ship during a storm.[23] Yet the court simultaneously acknowledged that the dock owner would have had to pay damages to the defendant if the dock owner had subsequently unmoored the defendant's ship, causing it to be damaged. *Vincent's* discussion of *Ploof v. Putnam* makes clear that the shipowner's option to take can itself be retaken if damages are paid:

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[21]Under the privilege of necessity, a defendant is permitted to commit an intentional tort to another's rights in property or realty to protect a more valuable interest in property or an interest in bodily security or life. *See* RESTATEMENT (SECOND) OF TORTS §§ 262, 263 & cmt. d (1965). Where the more valuable interest belongs to a large number of persons, for example, where a city must be saved from a fire, the privilege is one of public necessity, and the defendant owes no compensation. *See id.* § 262 & cmt. d. However, where the more valuable interest belongs only to the defendant or a small number of persons, the privilege is classified as a case of private necessity, and the defendant must still compensate the plaintiff for the harm caused by the invasion. *See id.* § 263(2) & cmt. e. Because compensation is owed, the privilege is said to be incomplete. However, because the defendant has a privilege, the plaintiff must pay for the damages caused by any self-help she undertakes to avoid the taking. *See id.* § 263 cmt. b; *see also* Ploof v. Putnam, 71 A. 188 (Vt. 1908).

[22]124 N.W. 221 (Minn. 1910).

[23]*See id.* at 222.

> In Ploof v. Putnam... the Supreme Court of Vermont held that where, under stress of weather, a vessel was without permission moored to a private dock at an island in Lake Champlain owned by the defendant, the plaintiff was not guilty of trespass, and that the defendant was responsible in damages because his representative upon the island unmoored the vessel, permitting it to drift upon the shore, with resultant injuries to it. If, in that case, the vessel had been permitted to remain, and the dock had suffered an injury, we believe the shipowner would have been held liable for the injury done.[24]

The shipowner's option–a liability rule–is itself protected by a liability rule.

Jon Hanson and Matt Stowe have identified *Vincent* as a vivid example of how the common law protects an option to take an entitlement (a liability rule) with another liability rule.[25] The dock owner holds the initial entitlement to the physical security of the dock. The shipowner (because of the exigencies of the storm) has a first-stage option to "take" the dock by mooring the ship to it and by paying damages for any injury that results. The dock owner has a second-stage option to unmoor the ship, but at a cost: The dock owner gives up a cause of action against the shipowner for damages and exposes himself to tort liability for any resulting damages to the ship and its crew. Exercising this second-stage option imposes on the dock owner a direct cost (potential tort liability) and an opportunity cost (potential tort damages).[26]

Our second example comes from Robert Ellickson. In the early 1970s, Ellickson proposed a modification of nuisance rules that would amount to a second-order liability rule, and this chapter has in

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[24] *Vincent*, 124 N.W. at 222. The dock owner's second-order option may have been limited. It might be that a Johnny-on-the-spot shipowner could have obtained an injunction to prevent the dock owner from unmooring the ship. But while the discussion in *Vincent* is dicta, there is no suggestion that the dock owner would have had to pay punitive damages for unmooring the ship. If a court is more likely to protect the shipowner's entitlement with an ex ante injunction, it would be more likely to deter takings with exemplary damages ex post any such bad faith unmooring.

[25] Hanson and Stowe refer to the *Vincent* standard as a "two-sided" liability rule. Jon Hanson & Matt Stowe, Lecture Notes, Torts, Harvard Law School (Fall 1996) (on file with the *Yale L. J.*).

[26] We emphasize this dual cost because readers are likely to imagine that the total cost of the dock owner's action is the payment of damages. It is important to account for these opportunity costs–foregoing damages created by the other party's previous taking–if we wish to understand how much exercising an option really costs an actor.

large part been inspired by his analysis.[27] Ellickson argued that when a landowner committed an intentional nuisance or other unneighborly activities, the landowner would be liable for damages, but that other parties could enjoin continuation of the activity if they were willing to compensate the landowner for any losses he suffered from that injunction.[28] Under Ellickson's proposed regime, the defendant (polluter) decides whether to purchase the right to pollute, and the plaintiff (pollutee) then decides whether to purchase an injunction to stop the pollution.[29]

2. Formalizing a sequence of call or put options: defining a higher-order rule regime

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[27]*See* Robert C. Ellickson, *Alternatives to Zoning: Covenants, Nuisance Rules, and Fines as Land Use Controls*, 40 U. CHI. L. REV. 681 (1973). In fact, Ellickson, Hanson, and Stowe, toourknowledge, are the only people who have seriously analyzed the potential utility of higher-order liability rules. In 1980, Mitch Polinsky saw that the law could give both polluters and pollutees a liability option to change the initial amount of legally permissible pollution. *See* A. Mitchell Polinsky, *Resolving Nuisance Disputes: The Simple Economics of Injunctive and Damage Remedies*, 32 STAN. L. REV. 1075, 1086-88 (1980). Polinsky opined that this type of regime "has not to our knowledge been considered by legal commentators or the courts. Since this remedy turns out to be unhelpful in most of the situations examined in this article, we will hereafter ignore it." *Id.* While Polinsky's article included a pathbreaking analysis of first-order liability rules, he never addressed the sequence in which second-order taking options might be exercised. *See* also Morris, *supra* note 45, at 822, 891-93 (recognizing possible usefulness of second-order liability rules, but not pursuing the question of when these rules might be efficient).

[28]*See* Ellickson, *supra* note 241, at 748. Ellickson described this proposal as a combination of two different types of entitlement regimes originally offered by Calabresi and Melamed. *See id.* at 738. Calabresi and Melamed's "Rule 2" gives the polluter an option to pollute and pay damages, while their "Rule 4" gives the pollutee an option to enjoin pollution by paying damages to the polluter. *See* Calabresi & Melamed, *supra* note 19, at 1115-24.

[29]In contrast, the "purchased injunction" featured in the famous case of *Spur Industries v. Del E. Webb Development Co*., 494 P.2d 700 (Ariz. 1972), represents a first-order liability rule. The polluter, in this case the owner of a feed lot, has the original entitlement to pollute. However, this entitlement is only protected by a liability rule. The neighbors have the option to stop pollution by paying damages and purchasing an injunction. Their taking is then protected by a property rule in the form of that injunction. *See id.* at 705-08.

A mortgagor's right of redemption provides yet another example of a second-order rule. Statutes in roughly half of the states give a mortgagor the option to buy back property *after* a foreclosure sale, by paying the foreclosure sale purchaser the foreclosure sale price. *See* Michael H. Schill, *An Economic Analysis of Mortgage or Protection Laws*, 77 VA. L. REV. 489, 495 (1991). The foreclosure sale is often an explicit auction–harnessing the private information of third parties–which allows a nonconsensual taking of the property from the mortgagor. *See id.* at 493. The statutory right of redemption, however, gives the mortgagor a take-back option, which allows the mortgagor to signal a higher (or equivalent) valuation of the property. The right of redemption might be viewed as a way to harness public and private information about the property's value, especially if temporary illiquidity prevents a mortgagor from signaling a high valuation at the time of the foreclosure sale.

This section describes and generalizes the notion of higher-order liability rules—which represent a sequence of alternating taking options (iterated call rules) or alternating giving options (iterated put rules).

Iterated Call: Imagine that one of the disputants (without loss of generality) called "plaintiff" gets the initial entitlement. In the first stage, the defendant gets the initial option of buying the asset for D_{Δ}^{1}. The plaintiff gets the option of preventing this transfer by paying D_{Π}^{1}. In the second stage, the defendant has the option of responding by increasing his bid to D_{Δ}^{2}, in response to which the plaintiff can again prevent transfer by increasing his offer to D_{Π}^{2}. This process continues for up to n stages, the n-stage game being fully described by the two sequences of damages:

$$D_{\Delta}^{1} < D_{\Delta}^{2} < \cdots < D_{\Delta}^{n}, \qquad \text{(I.1a)}$$
$$D_{\Pi}^{1} < D_{\Pi}^{2} < \cdots < D_{\Pi}^{n}. \qquad \text{(I.1b)}$$

Iterated Put: An alternative higher-order liability rule entails a potential sequence of being given put options. Again imagine that the plaintiff get the initial entitlement. The plaintiff also gets the initial option to forcefully transfer the asset to the defendant and to receive damages D_{Π}^{1}. The defendant can prevent transfer by paying D_{Δ}^{1}. In response, the plaintiff can lower the damages he is to receive to D_{Π}^{2}, and the defendant can, in turn, prevent transfer by paying D_{Δ}^{2}. The n-stage game is fully described by the two sequences of damages

$$D_{\Pi}^{1} > D_{\Pi}^{2} > \cdots > D_{\Pi}^{n}, \qquad \text{(I.2a)}$$
$$D_{\Delta}^{1} < D_{\Delta}^{2} < \cdots < D_{\Delta}^{n}. \qquad \text{(I.2b)}$$

Ayres and Goldbart showed that higher order rules could, by harnessing the private information of both litigants, produce first-best allocative efficiency. We now turn to the main focuses of the present Paper, viz., the development of instantaneous rules – that mimic the information harnessing of higher order rules without incurring the transaction costs of multiple takings.

II. Extension to continuous rules

The purpose of this section is to show that a larger family of liability rules exists that (a) containing the higher order rules of the previous section as simple special cases, and (b) are capable of achieving the same, first-best efficiency. In addition to formulating these rules, we shall explore a number of illustrative examples. In the following section we shall examine the new rules from a game-theoretic perspective; and in the concluding section of the Paper, we shall, among other things, draw analogies between these new rules and the familiar topic of auctions.

A. Continuous call rule

Let us examine more closely the *iterated call* rule. It will prove convenient to make a slight change of language here. We wish to refer to the exercising of an option as the *making of a bid,* and to the corresponding damages as the *amount of the bid.* Two factors distinguish this notion of bidding from the conventional one. First, the permitted bid amounts are drawn from a discrete set, which is specified by the court. And second, whereas in conventional bidding *any* bid must exceed the *opponent's* previous bid, in the present setting a litigant's bid must exceed only his *own* previous one. Stated mathematically, the set of interlaced conditions

$$0 < D_\Delta^1 < D_\Pi^1 < D_\Delta^2 < D_\Pi^2 < \cdots < D_\Delta^n < D_\Pi^n, \qquad \text{(II.1)}$$

is replaced by the two separate sets of conditions:[30]

$$0 < D_\Delta^1 < D_\Delta^2 < \cdots < D_\Delta^n; \qquad \text{(II.2a)}$$
$$0 < D_\Pi^1 < D_\Pi^2 < \cdots < D_\Pi^n. \qquad \text{(II.2b)}$$

Let us now make a natural generalization of this scheme. We take the limit $n \to \infty$, so that the increments of the bid amounts, $\{D_\Pi^{k+1} - D_\Pi^k\}$ and $\{D_\Delta^{k+1} - D_\Delta^k\}$ become infinitesimal quantities. It is convenient (and always possible) to view the damages $\{D_\Pi^k\}$ and $\{D_\Delta^k\}$ as the values of a pair of continuous *damages functions* $D_\Pi(s)$ and $D_\Delta(s)$, evaluated at the arguments $\{s^k \equiv (k-1)n\}$. In order to meet the conditions that the pair of discrete sequences of damages $\{D_\Pi^k\}$ and $\{D_\Delta^k\}$ each be strictly monotonically increasing, we shall require that the derivatives of the damages functions obey $D'_\Pi(s) > 0$, and $D'_\Delta(s) > 0$, where the prime indicates a derivative.

As another slight change of language, we shall refer to the *arguments* of the damages functions *$\{s^k\}$* as the *bids,* rather than the actual values of the damages functions at these arguments. We shall refer to the latter as *bid amounts*. As, in the continuous (i.e. $n \to \infty$) limit, the bid increments $s^{k+1} - s^k (= 1/n)$ tend to zero, the bids are drawn from the continuous interval [0; 1). The convenience of this

[30]We simply rewrite Eqs. (I.la) and (LLib).

focus on the bids *s* [rather than the bid amounts $D_{\Pi}(s)$ and $D_{\Delta}(s)$] lies in the intuitiveness of the notion that it is the highest bid that wins (i.e. ends up with the asset). To emphasize this point, consider a situation in which a plaintiff and a defendant respectively make bids s_{Π} and s_{Δ}, with $s_{\Pi} > s_{\Delta}$. Then the plaintiff's bid is the winning bid, even though it perfectly well may happen that $D_{\Pi}(s_{\Pi}) < D_{\Delta}(s_{\Delta})$.

In setting up the continuum generalization of the iterated call rule, we shall be introducing three conceptual steps. First, we propose that the procedure in which the explicit bids and counter-bids are made is replaced by one in which the bids (still drawn from the discrete set $\{s^k\}$) are announced by the court. Second, noting that bid increments tend to zero, we may assume that the current bid, represented by the number *s*, increases from zero to one, continuously with time. Third, we entirely eliminate the explicit bidding involved, by requiring the parties simply to secretly submit their final bids to the court.

In the scheme that follows from making the first step, the bids are announced sequentially, one by one, starting with the smallest bid of zero. At each step, there are three possible outcomes: (i) the defendant folds [i.e. allows the plaintiff to control the asset in exchange for the previously announced damages $D_{\Pi}(s^{k-1})$][31]; (ii) the defendant stays but the plaintiff folds [i.e. allows the defendant to control the asset in exchange for the damages $D_{\Delta}(s^k)$]; (iii) both defendant and plaintiff stay [i.e. that, to control the asset, the defendant is willing to pay the bid amount $D_{\Delta}(s^k)$ and the plaintiff is willing to pay the bid amount

[31] We may take $D_{\Pi}(s^o) = 0$, which means that the asset simply stays with the plaintiff if the defendant chooses not to exercise his first option.

$D_{\Pi}(s^k)]$, in which case the next bid is announced. If either (i) or (ii) is realized then the procedure stops, the asset is transferred accordingly, and the appropriate damages are paid.

Upon making the second step, in which we imagine letting the bid rise from zero to one, continuously in time, the litigants need only to indicate, at some time, their desire to fold. In the (unlikely) event of the litigants folding simultaneously, priority is given to the defendant (i.e. the defendant is taken to be the party who has folded first).

In the third, and final, step we observe that neither party has received any useful information until the bidding ends with one or other party being the winner. Thus, the parties know their maximum bids before bidding has commenced. And, thus, the entire bidding procedure can be dispensed with in favor of a procedure in which the parties simply furnish the court with their maximum bids. The court can use this information to immediately ascertain which party is the winner, as well as the amount of the winning bid.[32] The bidding stops when *s* reaches the smaller of the two secret bids, s_{Π} and s_{Δ}. Therefore, the party that made the highest bid is the winner. The damages are determined, however, by the *loser's* bid, as this is the bid at which the bidding stops.

It remains to mention that the restriction that *s* lie in the interval [0; 1) is quite arbitrary: the essentials of the problem are invariant with respect to arbitrary reparametrization. That is, for any monotonically increasing function of *s*, say *t*(*s*), one equivalently can work with functions $\widetilde{D}_{\Pi}(t) \equiv D_{\Pi}(t(s))$ and

[32]A popular web auction eBay (tm) has a system called *proxy bidding,* which lets users indicate their maximum bid (which is kept private), and simulates the bidding process by bidding incrementally on behalf of each user up to his maximum bid.

$\widetilde{D}_\Delta(t) \equiv D_\Delta(t(s))$. In the resulting scheme, the players should choose their bids to be $t_\Pi = t(s_\Pi)$ and $t_\Delta = t(s_\Delta)$, where s_Π and s_Δ would be their bids under the old scheme. As *t(s)* is monotonically increasing, the winner remains the same; the amount of damages paid, as determined by the lower bid, would similarly be unchanged. If one takes, for instance, $t(s) = t/)l - t)$, the domain of damages functions would be mapped from the interval [0; 1) into the entire positive real axis [0; $+\infty$).

The final version of the bidding procedure admits an appealing geometric interpretation, which we now explain. The damages functions $D_\Pi(s)$ and $D_\Delta(s)$ define, parametrically, a curve in the (D_Π, D_Δ) plane (see figure 1). The positivity of $D'_\Pi(s)$ and $D'_\Delta(s)$ implies that the tangent to the curve always points towards the positive quadrant (i.e. angles from 0º to 90º). The bids correspond to points on this curve, the higher bid being the point lying further along the curve (in the direction of increasing damages). The damages are determined by projecting the point for the losing bid on to the D_Π axis (if the plaintiff is the winner) or on to the D_Δ axis (if the defendant is the winner). This geometric reinterpretation illuminates the reparametrization freedom: the pairs $\{D_\Pi(s), D_\Delta(s)\}$ and $\{\widetilde{D}_\Pi(t), \widetilde{D}_\Delta(t)\}$ define the same curve.

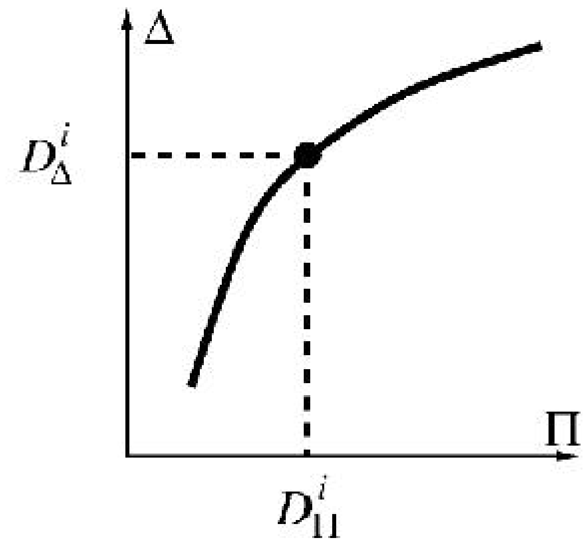

Fig. 1. An example of the parametrically defined bidding curve. The damages paid are obtained by projecting the lower bid onto Π- or Δ-axis.

B. Continuous put rule

We now examine the continuous put rule. Recall that the iterated put is characterized by the sets of damages $D_\Pi^1 > D_\Pi^2 > \cdots > D_\Pi^n$ and $D_\Delta^1 < D_\Delta^2 < \cdots < D_\Delta^n$.[33] We proceed to define damages functions $D_\Pi(s)$ and $D_\Delta(s)$ that satisfy $D_\Pi(s^k) = -D_\Delta^k$ and $D_\Delta^k(s^k) = D_\Pi^k$. This definition may seem peculiar, however it represents the fact that whoever ends up with the asset pays damages $D\ (s)$, where α stands for either Π or Δ . Indeed, if the plaintiff exercises his option, the defendant (Δ) ends up with the asset and pays $D_\Pi^k \equiv D_\Delta(s^k)$. If the defendant exercises his put-back option, the plaintiff receives the asset and D_Δ^k in damages (or, equivalently, pays the negative amount $-D_\Delta^k = D_\Pi(s^k)$ in damages). For the continuous put rule we prefer to define $s^k \equiv (n-1-k)/n$ so that one still has $D_\Pi'(s) > 0$ and $D_\Delta'(s) > 0$, but now, during the bidding, s *decreases* from 1 to 0. We now consider the last (and somewhat tricky step: in contrast to the continuous put case, the asset ends up in the hands of the party who *refused* to continue the bidding, i.e., did not exercise his put option.

In constructing the generalized continuous put rule, we again make three revisions of the bidding game. In the first revision, the bids are announced by the court, and the litigants announce whether they are willing to exercise their put options. Should one of them refuse, he becomes the owner of the asset and pays damages determined by the current bid. If both refuse, priority is given to the initial asset holder (the plaintiff). The second revision lets the bid decrease continuously from 1 to 0, until one of the litigants

[33] See Eqs. (1.2a) and (1.2b).

announces his intention to *take* the asset. The damages are determined by the current bid. In the final revision, the litigants submit their secret *taking bids.* The court then assigns the asset to the party with the higher bid. The damages are determined by the *winning* bid, unlike the situation in continuous call rule, in which the losing bid is used to calculate the damages. The difference between the continuous call and the continuous put rules is that the former favors strategic overbidding whereas the latter favors strategic underbidding. The continuous put also admits a geometric interpretation: the winning bid (the point further along the curve determines the winner, and its projection onto either the D_Π or the D_Δ axis determines the damages

C. Determining optimal strategies

The model that we consider throughout the Paper is the familiar one in which the plaintiff and defendant each possess an element of private information —the value that they place on a right (or *asset,* as we shall commonly refer to it) that is under dispute. These private valuations are denoted V_Π and V_Δ for the plaintiff and defendant, respectively; we assume that they are random variables[34] distributed according to a joint probability distribution (j.p.d.) which we denote $f(V_\Pi, V_\Delta)$. This distribution is assumed to be public

[34] This assumption should not be taken too literally. It simply means that the public has incomplete information, and that a probability distribution is being used to represent their beliefs.

knowledge:[35] it is known to the plaintiff, to the defendant, and to the court. We shall say that the private valuations are uncorrelated if the j.p.d. is factorizable, i.e., if $f(V_\Pi, V_\Delta) = f_\Pi(V_\Pi) f_\Delta(V_\Delta)$). Otherwise, we shall say that the private valuations are correlated.

For the correlated case, we shall also make use of the conditional probability densities: $f_\Pi(V_\Pi | V_\Delta \equiv f_\Pi(V_\Pi) f_\Delta(V_\Delta)$ and similarly for $f_\Delta(V_\Delta | V_\Pi)$.

1. Continuous call rule

In the present section we assume that s_Π and s_Δ are, respectively, the bids submitted by the plaintiff and the defendant (referred to in this section as the players Π and Δ . In addition to conditional probability distributions $f_\Pi(V_\Pi | V_\Delta)$ and $f_\Delta(V_\Delta | V_\Pi)$, we shall make use of cumulative distributions, denoted by symbol *F*, e.g.,

$$F_\Delta(V_\Delta\ V_\Pi) = \int_0^{V_\Delta} f_\Delta(v|V_\Pi)\, dv. \qquad \text{(II.3)}$$

Now, each player is faced with the problem of making the optimal bid, given his private valuation of the asset. In doing so, he must have some knowledge of other player's intentions, i.e., other player's strategy. Therefore, we should end up with a pair of coupled equations for the strategies. Also note that

[35]This is what is often meant by *first-order beliefs*. More precisely, first-order beliefs correspond to the uncorrelated case, i.e. f_Π and $f_\Delta(V_\Delta)$ being separately known. The correlated j.p.d. cannot be described by first-order beliefs alone due to the fact that e.g. defendant's belief about the distribution of plaintiff's valuations depends on his own valuation, and, therefore is not known to others. The correlated example is the simplest one to go beyond first-order beliefs; and yet it renders calculations analytically tractable.

in contrast to games with complete information, in which the strategy is essentially a number (the actual bid), under incomplete information the *strategy* should be defined as a function that maps players' internal valuations into bids: $s_\Pi(V_\Pi)$ and $s_\Delta(V_\Delta)$.[36] We shall, in fact, work with the inverses of these functions, viz., $V_\Pi(s)$ and $V_\Delta(s)$.[37] We shall also use the latter to define the distributions $\hat{f}(s) \equiv f_\Pi(V_\Pi(s))V_\Pi'(s)$ and $\hat{f}_\Delta(s) \equiv f_\Delta(V_\Delta(s))V_\Delta'(s)$ which are then the probability distributions of the actual bids. We also define the corresponding conditional probability distributions $\hat{F}_\Pi(s|V_\Delta)$ and $\hat{f}_\Delta(s|V_\Pi)$ and their cumulative versions, $\hat{F}_\Pi(s|V_\Delta)$ and $\hat{F}_\Delta(s|V_\Pi)$.

If Π's internal valuation is V_Π, his bid $s_\Pi(V_\Pi)$ is determined so as to maximize his payoff. In this maximization he assumes that Δ follows his optimal strategy $s_\Delta(V_\Delta)$. As Π does not know Δ's private valuation V_Δ, he maximizes his expected payoff under the condition that V_Δ is randomly distributed according to $f_\Delta(V_\Delta|V_\Pi)$. Then Δ's private valuation does not influence Π's payoff, except through Δ's bid. Equivalently, Π works with the distribution of Δ's bids, $\hat{f}_\Delta(s|V_\Pi)$.

We now proceed to implement the determination of optimal strategies outlined in the previous paragraph. After the plaintiff and the defendant have submitted their secret bids, s_Π and s_Δ, the asset goes the higher bidder at damages determined by the lower bidder. Therefore, the plaintiff's and the defendant's

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[36]We thus ignore the possibility of mixed strategies.

[37]The inverses exist as long as $s_\Pi(V_\Pi)$ and $s_\Delta(V_\Delta)$ are monotonic, and we shall assume that they are.

respective payoffs, π_Π and π_Δ, are given by

$$\pi_\Pi(s_\Pi, s_\Delta | V_\Pi, V_\Delta) = \begin{cases} V_\Pi - D_\Pi(s_\Delta), & \text{if } s_\Pi > s_\Delta, \\ D_\Delta(s_\Pi), & \text{if } s_\Pi < s_\Delta, \end{cases} \quad \text{(II.4a)}$$

$$\pi_\Delta(s_\Pi, s_\Delta | V_\Pi, V_\Delta) = \begin{cases} D_\Pi(s_\Delta), & \text{if } s_\Pi > s_\Delta, \\ V_\Delta - D_\Delta(s_\Pi), & \text{if } s_\Pi < s_\Delta. \end{cases} \quad \text{(II.4b)}$$

Taking into account the probabilities of private valuations, we can write down the expressions for the expected payoffs:

$$\pi_\Pi(s_\Pi | V_\Pi) = \int_0^{s_\Pi} ds_\Delta \, [V_\Pi - D_\Pi(s_\Delta)] \hat{f}_\Delta(s_\Delta | V_\Pi) + D_\Delta(s_\Pi)[1 - \hat{F}_\Delta(s_\Pi | V_\Pi)], \quad \text{(II.5a)}$$

$$\pi_\Delta(s_\Delta | V_\Delta) = \int_0^{s_\Delta} ds_\Pi \, [V_\Delta - D_\Delta(s_\Pi)] \hat{f}_\Pi(s_\Pi | V_\Delta) + D_\Pi(s_\Delta)[1 - \hat{F}_\Pi(s_\Delta | V_\Delta)]. \quad \text{(II.5b)}$$

Here, $\pi_\Pi(s_\Pi | V_\Pi)$ has the meaning of Π's expected payoff, given that his valuation is V_Π and his bid is s_Π; and similarly for $\pi_\Delta(s_\Delta | V_\Delta)$. The players follow their optimal strategies, i.e., they maximize their individual payoffs with respect to their individual bids. Thus, the bids obey the stationarity condition:

$$\frac{d}{ds_\Pi} \pi_\Pi(s_\Pi | V_\Pi) = \frac{d}{ds_\Delta} \pi_\Delta(s_\Delta | V_\Delta) = 0. \quad \text{(II.6)}$$

Inserting the explicit expressions for the expected payoffs Eqs. (II.5a) and (II.5b) leads to the conditions:

$$\left[V_\Pi\left(s_\Pi\right) - D_\Pi\left(s_\Pi\right)\right]\hat{f}_\Delta\left(s_\Pi \middle| V_\Pi\right) = D'\left(s_\Pi\right)\left[1 - \hat{F}_\Delta\left(s_\Pi \middle| V_\Pi\right)\right] - D_\Delta\left(s_\Pi\right)\hat{f}_\Delta\left(s_\Pi \middle| V_\Pi\right) = 0 \,, \quad \text{(II..7a)}$$

$$\left[V_\Delta\left(s_\Delta\right) - D_\Delta\left(s_\Delta\right)\right]\hat{f}_\Pi\left(s_\Delta \middle| V_\Delta\right) + D'_\Pi\left(s_\Delta\right)\left[1 - \hat{F}_\Pi\left(s_\Delta \middle| V_\Delta\right)\right] - D_\Pi\left(s_\Pi\right)\hat{f}_\Pi\left(s_\Delta \middle| V_\Delta\right) = 0 \,, \quad \text{(II.7b)}$$

which, upon rearrangement, may be written as the following set of conditions:

$$D'_\Pi(s) = \lambda_\Pi(s|V_\Delta(s))[D_\Pi(s) - D_\Delta(s) - V_\Delta(s)], \qquad \text{(II.8a)}$$
$$D'_\Delta(s) = \lambda_\Delta(s|V_\Pi(s))[D_\Pi(s) \quad D_\Delta(s) \quad V_\Pi(s)], \qquad \text{(II.8b)}$$

the coefficients $_\Pi$ and $_\Delta$ in the conditions being defined via

$$\lambda_\Pi(s|V_\Delta) \equiv \hat{f}_\Pi(s|V_\Delta) \Big/ [1 - \hat{F}_\Pi(s|V_\Delta)], \qquad \text{(II.8c)}$$
$$\lambda_\Delta(s|V_\Pi) \equiv \hat{f}_\Delta(s|V_\Pi) \Big/ [1 - \hat{F}_\Delta(s|V_\Pi)], \qquad \text{(II.8d)}$$

In principle, conditions (II.8c) and (II.8d) should, for a given pair of damages functions $D_\Pi(s)$ and $D_\Delta(s)$, be solved for the (inverse) bidding strategy functions. This appears to be a formidable task, as the equations are, in general, non-linear [the non-linearity coming from the dependence of X_Π and X_Δ on $V_\Pi(s)$ and $V_\Delta(s)$]. However, for reasons that we shall explain in section III, a less demanding route is available and, in fact, appropriate.

2. Continuous put rule

We now examine the continuous put rule. The only difference from the continuous call case is that the damages are determined by the winning bid. Accordingly, we have the expected payoffs:

$$\pi_\Pi(s_\Pi|V_\Pi) = [V_\Pi - D_\Pi(s_\Pi)]\hat{F}_\Delta(s_\Pi|V_\Pi) + \int_{s_\Pi}^{\infty} ds_\Delta\, D_\Delta(s_\Delta)\hat{f}_\Delta(s_\Delta|V_\Pi), \qquad \text{(II.9a)}$$

$$\pi_\Delta(s_\Delta|V_\Delta) = [V_\Delta - D_\Delta(s_\Delta)]\hat{F}_\Pi(s_\Delta|V_\Delta) + \int_{s_\Delta}^{\infty} ds_\Pi\, D_\Pi(s_\Pi)\hat{f}_\Pi(s_\Pi|V_\Delta), \qquad \text{(II.9b)}$$

and, again, construct stationarity conditions: $(d/ds_\Pi)\,\pi_\Pi(s_\Pi|V_\Pi) = (d/ds_\Delta)\,\pi_\Delta(s_\Delta|V_\Delta) = 0$. By substituting the explicit expressions for the expected payoffs, stationarity conditions can be written in the following form:

$$-D'_\Pi(s_\Pi)\hat{F}_\Delta(s_\Pi|V_\Pi) + [V_\Pi(s_\Pi) - D_\Pi(s_\Pi) - D_\Delta(s_\Pi)]\hat{f}_\Delta(s_\Pi|V_\Pi) = 0, \qquad \text{(II.10a)}$$

$$-D'(s_\Delta)\hat{F}_\Pi(s_\Delta|V_\Delta) + [V_\Delta(s_\Delta) - D_\Delta(s_\Delta) - D_\Delta(s_\Delta)]\hat{f}_\Pi(s_\Delta|V_\Delta) = 0 \qquad \text{(II.10b)}$$

We can recast this conditions into a form similar to that of Eqs. (II.8a) and (II.8b),

$$D'_\Pi(s) = \mu_\Delta(s|V_\Pi(s))[V_\Pi(s) - D_\Pi(s) - D_\Delta(s)], \qquad \text{(II.11a)}$$

$$D'_\Pi(s) = \mu_\Pi(s|V_\Delta(s))[V_\Delta(s) - D_\Pi(s) - D_\Delta(s)], \qquad \text{(II.11b)}$$

where we have introduced

$$\mu_\Pi(s|V_\Delta) \equiv \hat{f}_\Pi(s|V_\Delta) \Big/ \hat{\bar{F}}_\Pi(s|V_\Delta), \tag{II.11c}$$

$$\mu_\Delta(s|V_\Pi) \equiv \hat{f}_\Delta(s|V_\Pi) \Big/ \hat{\bar{F}}_\Delta(s|V_\Pi). \tag{II.11d}$$

D. Recovering the discrete rules

We have seen that when the asset-allocation decision is delegated to the litigants the economic efficiency of the allocation is increased by a suitable choice of the damages in a vanilla call or put regime. Work on exotic liability rules has been sparked by the sense that efficiency would be further increased if the litigants were to have the greater freedom offered by iterated call or put regimes, and the court were to have at its disposal the correspondingly greater number of adjustable damages parameters — infinitely many in the continuous call and put cases.

Let us put these arguments on a firmer basis. It has been argued that the vanilla call is, in general, more efficient than the property rule, as the property rule can be thought of as vanilla call with damages *D* set to infinity. Barring the exceptional possibility that the total efficiency is independent of *D*, a liability call rule can always be made more efficient than a property rule by a better choice of *D*. As we shall now explain, the continuous call rule is more efficient than the iterated call, provided that the iterated call can be shown to follow from the continuous version with appropriately chosen parameters. This is indeed possible if the damages curve of the continuous version is chosen to have a zigzag shape, as shown in figure 2.

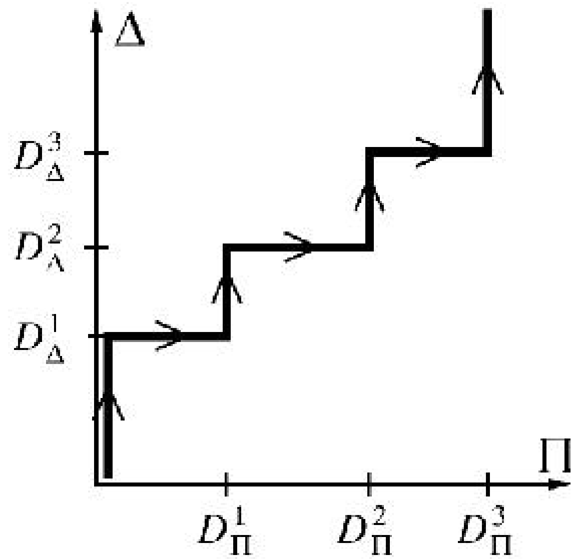

Fig. 2.— A continuous rule bidding curve that simulates a 3-stage iterated call rule.

With this zigzag shaped damages curve, the plaintiff should only bid at corners at which the zigzag curve turns right, and the defendant should only bid at corners where the curve turns left (or at the point at the origin, or infinitely far away). To be precise, we will show that if one of the players would use the strategies from this subgame (i.e. bidding only in corners), the other player gains no advantage by deviating from the subgame, i.e. by bidding away from corners. Indeed, if the defendant decides to place a bid somewhere on a horizontal segment, he would simply increase the amount of damages he receives if he loses by moving his bid point to the right. If he decides to bid somewhere on a vertical segment, he is indifferent where to place his bid, but it is safer to move down, thereby decreasing the damages he might have to pay (without changing the damages he might receive), should the plaintiff decide to play away from corner. This shows the sought reduction from the continuous call to the discrete iterated call. Similar reasoning can be used to relate the continuous put and the discrete iterated put rules. We refer the reader to the A for a table of damage functions used to simulate each of the liability rules described in the Introduction.

E. Connections with auctions

The procedure we have described in this chapter has much in common with familiar auctions. In the case of auctions, the relative of our iterated call is the increasing price, or English, auction. The iterated put, in turn, corresponds to the decreasing price, or Dutch, auction. Arguments similar to the ones we have made (in going from the infinitesimal bid increments to "sealed envelope" bidding) led Vickrey[38] to propose replacing English auctions with the so-called Vickrey, or second-best, auction, and to replace the Dutch auction with the first-best auction. In both Vickrey and first-best auctions, the bidders secretly submit their bids. The highest bid becomes the winner at the price determined by the second-highest (Vickrey) or the highest (first-best bid. The analogy ends here, because in our formulation, the damages paid, do not, in

[38] William Vickrey, *Counterspeculation, Auctions and Competitive Sealed Tenders*, 16 J. Finance 8 (1961).

general, correspond to the real bids. The importance of the Vickrey auction is in that the bidders bid according to their private valuations, whereas in first-best auctions, they tend to strategically underbid. In contrast, in our scheme, the bid amounts under the iterated put rule reflects strategic underbidding, whilst strategic overbidding features under the iterated call rule.

III. Designing Optimal Mechanism

A. Formulation

The task of designing an optimal mechanism can be split into two parts. The first part involves finding optimal bidding strategies given a set of damage parameters. The second part involves maximizing total expected efficiency by adjusting these parameters. For the continuous call or put rules the strategy is synonymous with submitting a bid, and the set of parameters corresponds to the damages functions $D_{\Pi}(s)$ and $D_{\Delta}(s)$. The problem of determining the optimal bid, given $D_{\Pi}(s)$ and $D_{\Delta}(s)$, was addressed and solved (at least formally) in the previous section. Approaching the second part of the problem (i.e. the maximization of total expected utility), we note that for the *first-best allocation* (i.e. a mechanism whereby the party with the higher private valuation always gains control of the asset ~ the total expected efficiency achieves its optimal value. As we shall now see, a model in which the set of parameters corresponds to the pair of continuous functions is sufficiently rich to achieve first-best allocational efficiency. Any other mechanism will be, at best, only as efficient, not more so.

Under both the continuous call and the continuous put rules, the asset is allocated to the party with the higher bid, i.e. to the plaintiff if $s_{\Pi} > s_{\Delta}$ so and to the defendant if $s_{\Pi} < s_{\Delta}$.On the other hand, for optimal allocation we must have that the asset is allocated to the plaintiff if $V_{\Pi}(s_{\Pi}) > V_{\Delta}(s_{\Delta})$ and to the defendant if $V_{\Pi}(s_{\Pi}) < V_{\Delta}(s_{\Delta})$. A moment's reflection will show that these conditions are compatible if and only if $V_{\Pi}(s) = V_{\Delta}(s)$. On the other hand, we observed that the game is invariant[39] with respect to the reparametrization of $D_{\Pi}(s)$ and $D_{\Delta}(s)$ via any monotonically increasing function *t(s)*. Assuming that there exists a mechanism [i.e. functions $D_{\Pi}(s)$ and $D_{\Delta}(s)$] that guarantee $V_{\Pi}(s) = V_{\Delta}(s)$, by choosing $t = V_{\Pi}(s) = V_{\Delta}(s)$ as the new parameter, the mechanism can be reformulated via $D_{\Pi}(t)$ and $D_{\Delta}(t)$ so that $V_{\Pi}(t) = V_{\Delta}(t) = t$.

We now substitute $V_{\Pi}(s) = s$ and $V_{\Delta}(s) = s$ into the equations Eqs. (II.8a) and (II.8b) of section C for the optimal bids so that $D_{\Pi}(s)$ and $D_{\Delta}(s)$ are viewed as the unknowns rather than parameters:

Call	*Put*
$D'_{\Pi}(s) = {}_{\Pi}(s\vert V_{\Delta} = s)\left[D_{\Pi}(s) = D_{\Delta}(s) - s\right]$	$D'_{\Pi}(s) = {}_{\Delta}(s\vert V_{\Pi} = s)\left[s - D_{\Pi}(s) - D_{\Delta}(s)\right]$
$D'_{\Delta}(s) = {}_{\Delta}(s\vert V_{\Pi} = s)\left[D_{\Pi}(s) + D_{\Delta}(s) - s\right]$	$D'_{\Delta}(s) = {}_{\Pi}(s\vert V_{\Delta} = s)\left[s - D_{\Pi}(s) - D_{\Delta}(s)\right]$

Table 2: Differential equations for damages functions.

[39]To be precise, the term *covariant* should be used, as the calculated bids should be adjusted according to the reparametrization.

Litigants, furnished with $D_\Pi(s), D_{\Delta(s)}$ that solve these equations, would apply the rationale of section C, and in so doing would discover that their optimal strategies are $V_\Pi(s) = V_\Delta(s) = s$. Therefore, in making their bids, they would be forced to reveal their private valuations. By virtue of the fact that the asset is allocated to the higher bidder, first-best allocational efficiency is realized.

A special note should be made about the boundary conditions obeyed by the damages functions. To ensure that the equations of Table 2 have non-singular solutions, the following conditions must be met[40]:

Call	*Put*
$D_\Pi(s_{\max}) + D_\Delta(s_{\max}) = s_{\max}$	$D_\Pi(s_{\min}) + D_\Delta(s_{\min}) = s_{\min}$

Table 3: Boundary conditions for damages functions.

We now address the task of actually solving for the damages functions, given the optimal strategies $V_\Pi(s) = V_\Delta(s) = s$. To do this, we first introduce the auxiliary function $\widetilde{D}(s) \equiv D_\Pi(s) + D_\Delta(s)$. As seen by adding together the differential equations of Table 2, $\widetilde{D}(s)$ obeys a certain ordinary differential equation, depending on whether we are considering the continuous call rule or the continuous put rule:

Call	*Put*

[40] We introduce the following notation: $s_{\max}$ is the smallest s such that $f(t_\Pi, t_\Delta) = 0$ for all $t_\Pi, t_\Delta > s$; similarly, $s_{\min}$ the largest s such that $f(t_\Pi, t_\Delta) = 0$ for all $t_\Pi, t_\Delta < s$.

$\widetilde{D}'(s) = \quad (s)\left[\widetilde{D}(s) - s\right]$	$\widetilde{D}'(s) = \quad (s)\left[s - \widetilde{D}(s)\right]$
$\widetilde{D}(s_{\max}) = s_{\max}$	$\widetilde{D}(s_{\min}) = s_{\min}$

Table 4: Equations for $\widetilde{D}(s)$ together with boundary conditions.

where $(s) \equiv \; _{\Pi}(s|V_{\Delta} = s) + \; _{\Delta}(s|V_{\Pi} = s)$ and $(s) \equiv \; _{\Pi}(s|V_{\Delta} = s) = \; _{\Delta}(s|V_{\Pi} = s$; the quantities $_{\Pi}, \; _{\Delta}, \; _{\Pi}, \; _{\Delta}$ have been defined in section C. The stated boundary conditions on $\widetilde{D}$ follow from those given above for $D_{\Pi}(s)$ and $D_{\Delta}(s)$. Note that, owing to the linear, first-order form of the equation obeyed by $\widetilde{D}$, the method of integrating factors presents us with an explicit solution in terms of the functions (s) or (s), which encode information about the joint probability distribution $f(V_{\Pi}, V_{\Delta})$. Note that this distribution may be arbitrarily correlated.

Armed with the solution for $\widetilde{D}(s)$, we may return to one or other of the differential equations, for $D_{\Pi}(s)$ or for $D_{\Delta}(s)$, eliminate the sum $D_{\Pi}(s) = D_{\Delta}(s)$ on the right hand side in favor of $\widetilde{D}(s)$, and integrate to obtain $D_{\Pi}(s)$ and $D_{\Delta}(s)$.

Note, that $D_{\Pi}(s)$ and $D_{\Delta}(s)$ are determined only up to a single constant of integration, as they should be. If a pair $\left\{D_{\Pi}(s), D_{\Delta}(s)\right\}$ is a solution, then so is the pair $\left\{D_{\Pi}(s) + A, D_{\Delta}(s) - A\right\}$.

As in Ayres & Goldbart (20xx) it is possible for courts to decouple distributional and allocative concerns in that there is a family of allocatively equivalent damage curves that vary how the total expected value is divided between the plaintiff and defendant. In what follows, this constant of integration can be thought of as a free variable that a lawmaker can set to independently pursue equitable goals or to enhance ex ante investment incentives.

B. Examples

In this section we shall consider some elementary but, we hope, instructive examples involving the continuous call and continuous put rules. We shall take the joint probability distribution density to be constant (i.e. uniform distribution) throughout some geometrical region in the (V_Π, V_Δ) plane. We remind the reader that a rectangular region having sides parallel to the coordinate axes corresponds to an example of uncorrelated distribution[41], whereas an arbitrary shape implies correlations.

Uniform distributions lead to particularly simple expressions for the functions $_\Pi(s)$ and $_\Delta(s)$, as we shall now see. We refer to figure 3 for further discussion of this fact. There, the point *S* corresponds to $V_\Pi = V_\Delta = s$. Points, where the edge of the area is intersected by the upward and rightward rays drawn from *S* are labeled *U* and *R,* respectively. Then the functions $_\Pi(s) = f(s,s) \Big/ \int_s^\infty f(s,t)dt$ assume the following simple forms: $_\Pi(s) = 1/|SR|$ and $_\Delta(s) = 1/|SU|$, where $|AB|$ denotes the

[41]Indeed, for the rectangular region $f(V_\Pi, V_\Delta) = f_\Pi(V_\Pi) f_\Delta(V_\Delta)$, with $f_\Pi(V_\Pi) = 1/(V_\Pi^{\max} - V_\Pi^{\min})$ for $V_\Pi \in [V_\Pi^{\min}; V_\Pi^{\max}]$ and $f_\Delta(V_\Delta) = 1/(V_\Delta^{\max} - V_\Delta^{\min})$ for $V_\Delta \in [V_\Delta^{\min}; V_\Delta^{\max}]$, is constant everywhere in the rectangle having corners $(V_\Pi^{\min}, V_\Delta^{\min}), (V_\Pi^{\min}, V_\Delta^{\max}), (V_\Pi^{\max} V_\Delta^{\max}), (V_\Pi^{\max}, V_\Delta^{\min})$ and 0 elsewhere.

distance between points A and B. Similarly, for the functions $_{\Pi}(s)$ and $_{\Delta}(s)$ that appear in the continuous put case, we obtain (see figure 4): $_{\Pi}(s) = 1/|SL|$ and $_{\Delta}(s) = 1/|SB|$. We now use these results to compute damages functions in various example settings.

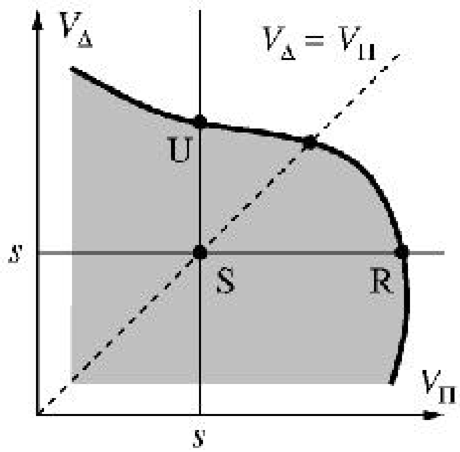

Fig. 3.— Determining $\lambda_{\Pi}(s)$ and $\lambda_{\Delta}(s)$ for a uniform distribution.

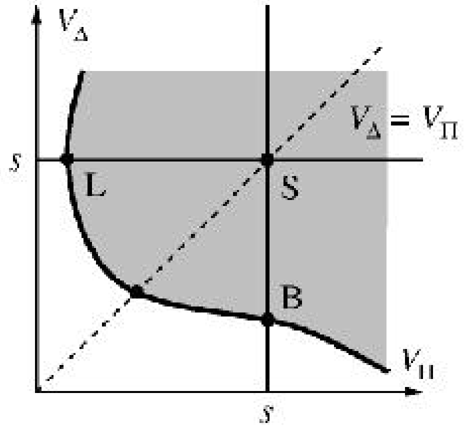

Fig. 4.— Determining $\mu_\Pi(s)$ and $\mu_\Delta(s)$ for a uniform distribution.

1. Uncorrelated uniform distribution: continuous call rule

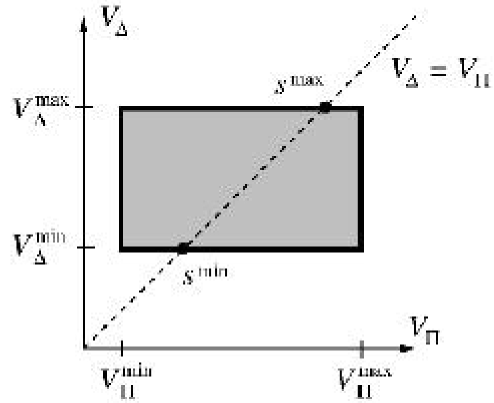

Fig. 5. Uniform distribution in a rectangle.

Let us take the joint probability distribution to be uniform in the rectangle $V_\Pi \in \left[V_\Pi^{\min}; V_\Pi^{\max}\right], V_\Delta \in \left[V_\Delta^{\min}; V_\Delta^{\max}\right]$. For this case, we have $_\Pi(s) = 1/\left(V_\Pi^{\max} - s\right)$ and $_\Delta(s) = 1/\left(V_\Delta^{\max} - s\right)$. For the sake of concreteness, let us further assume that $V_\Pi^{\max} > V_\Delta^{\max}$. In this case, the boundary condition (see section A) is enforced at $s_{\max} = V_\Delta^{\max}$. To solve for $\widetilde{D}(s)$, we multiply the equation it obeys by the integrating factor $\left(V_\Pi^{\max} - s\right)\left(V_\Delta^{\max} - s\right)$, thus obtaining:

$$\left(V_\Pi^{\max} - s\right)\left(V_\Delta^{\max} - s\right)\widetilde{D}'(s) = \left(V_\Pi^{\max} = V_\Delta^{\max} - 2s\right)\left(\widetilde{D}(s) - s\right) \tag{III.1}$$

Then, transferring the term proportional to $\widetilde{D}(s)$ to the left hand side and integrating gives:

$$\left(V_\Pi^{\max} - 2\right)\left(V_\Delta^{\max} - s\right)\widetilde{D}(s) = \frac{2}{3}s^3 - \frac{1}{2}\left(V_\Pi^{\max}\right)s^2 + K, \tag{III.2}$$

where the constant of integration K is fixed by the boundary condition that $\widetilde{D}(s)$ remain finite at $s = V_\Delta^{\max}$. At that value of s, both sides of the equation must vanish. Straightforward algebra then yields the following expression for $\widetilde{D}(s)$:

$$\widetilde{D}(s) = \frac{1}{6}\left(V_\Pi^{\max} = V_\Delta^{\max}\right) + \frac{2}{3}s - \frac{1}{6}\frac{\left(V_\Pi^{\max} - V_\Delta^{\max}\right)^2}{V_\Pi^{\max} - s} \tag{III.3}$$

which we remind the reader holds for the $V_\Pi^{\max} > V_\Delta^{\max}$ case of the uniform rectangular distribution. Inserting this solution for $\widetilde{D}(s)$ into the equation obeyed by $D_\Pi(s)$ gives:

$$D_\Pi'(s) = \frac{1}{3} - \frac{1}{6}\frac{V_\Pi^{\max} - V_\Delta^{\max}}{V\Pi^{\max} - s} - \frac{1}{6}\frac{\left(V_\Pi^{\max} - V_\Delta^{\max}\right)^2}{\left(V_\Pi^{\max} - s\right)^2}\ , \tag{III.4}$$

which may be straightforwardly integrated. Thus, one obtains $D_\Pi(s)$ and, using the relation

$D_\Delta(s) = \widetilde{D}(s) - D_\Pi(s)$, also $D_\Delta(s)$:

$$D_\Pi(s) \qquad \frac{1}{3}s + \frac{1}{6}(V_\Pi^{\max} - V_\Delta^{\max})\ln(V_\Pi^{\max} - s) - \frac{1}{6}\frac{(V_\Pi^{\max} - V_\Delta^{\max})^2}{(V_\Pi^{\max} - s)} + A, \qquad \text{(III.5a)}$$

$$D_\Delta(s) = \frac{1}{3}s - \frac{1}{6}(V_\Pi^{\max} - V_\Delta^{\max})\ln(V_\Pi^{\max} - s) + \frac{1}{6}(V_\Pi^{\max} - V_\Delta^{\max}) - A, \qquad \text{(III.5b)}$$

where A is the constant of integration. The resulting damages curve is shown in figure 6:

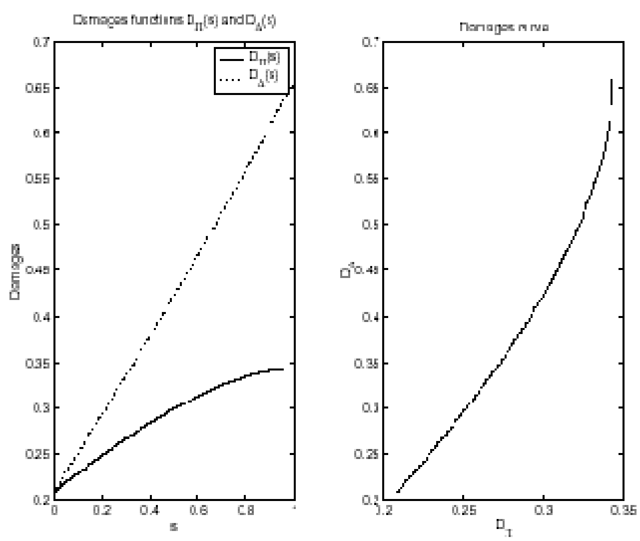

Fig. 6.— Damages obtained by considering the distribution uniform in a rectangle [0; 2] × [0; 1]. Left: Damages curve in the $D_\Pi - D_\Delta$ plane. Right: Actual damages functions $D_\Pi(s)$ and $D_\Delta(s)$.

The strategy just used also provides damages curves for the case $V_{\Pi}^{\max} < V_{\Delta}^{\max}$, in which case we obtain

$$D_{\Pi}(s) = \frac{1}{3}s + \frac{1}{6}(V_{\Delta}^{\max} - V_{\Pi}^{\max})\ln(V_{\Delta}^{\max} - s) - \frac{1}{6}\frac{(V_{\Delta}^{\max} - V_{\Pi}^{\max})^2}{(V_{\Delta}^{\max} - s)} + A, \qquad \text{(III.6a)}$$

$$D_{\Delta}(s) = \frac{1}{3}s - \frac{1}{6}(V_{\Delta}^{\max} - V_{\Pi}^{\max})\ln(V_{\Delta}^{\max} - s) + \frac{1}{6}(V_{\Delta}^{\max} + V_{\Pi}^{\max}) - A, \qquad \text{(III.6b)}$$

as well as for the case $V_{\Pi}^{\max} = V_{\Delta}^{\max}$, for which we obtain

$$D_{\Pi,\Delta}(s) = \frac{1}{6}V^{\max} + \frac{1}{3}s \pm A\,. \qquad \text{(III.7)}$$

We note that especially simple damage curves (i.e. they obey $D_{\Pi} - D_{\Delta} = const$) result for symmetrical problems, by which we mean settings for which $_{\Pi}(s) = {}_{\Delta}(s)$ [or, in the put-rule case, $_{\Pi}(s) = {}_{\Delta}(s)$]. These situations commonly arise for probability distributions that are symmetric with respect to the interchange of players: $f(V_{\Pi}, V_{\Delta}) = f(V_{\Delta}, V_{\Pi})$. Also note that setting $V_{\Pi}^{\min} = V_{\Delta}^{\min} = 0$ and $V_{\Pi}^{\max} = V_{\Delta}^{\max} = 1$, would correspond to

the special case addressed by Ayres and Balkin.[42]

[42]See *supra* note 6.

2. Uncorrelated uniform distribution: continuous put rule

Let us consider the same geometry as we did in the previous example, i.e., a uniform rectangular distribution. Now, however, for the sake of concreteness, let us assume that $V_{\Pi}^{\min} < V_{\Delta}^{\min}$, in which case the appropriate boundary condition is enforced at $s_{\min} = V_{\Delta}^{\min}$. Again we multiply the equation of Table 4 by a suitable integrating factor, thus obtaining

$$(s - V_{\Pi}^{\min})(s - V_{\Delta}^{\min})\tilde{D}'(s) = (2s - V_{\Pi}^{\min} - V_{\Delta}^{\min})[s - \tilde{D}(s)]. \qquad \text{(III.8)}$$

Integrating and applying the boundary condition gives

$$\tilde{D}(s) = \frac{1}{6}(V_{\Delta}^{\min} + V_{\Pi}^{\min}) + \frac{2}{3}s + \frac{1}{6}\frac{(V_{\Delta}^{\min} - V_{\Pi}^{\min})^2}{s - V_{\Pi}^{\min}}. \qquad \text{(III.9)}$$

Then, substituting this result into the equation for $D_{\Delta}(s)$ gives

$$D_{\Delta}'(s) = \frac{1}{s - V_{\Pi}^{\min}}\left[\frac{1}{3}s - \frac{1}{6}\left(V_{\Delta}^{\min} + V_{\Pi}^{\min}\right) - \frac{1}{6}\frac{\left(V_{\Delta}^{\min} - V_{\Pi}^{\min}\right)^2}{s - V_{\Pi}^{\min}}\right] \qquad \text{(III.10)}$$

and integrating and using $D_{\Pi}(s) = \tilde{D}(s) - D_{\Delta}(s)$ gives the damages functions:

$$D_{\Pi}(s) = \frac{1}{3} + \frac{1}{6}\left(V_{\Delta}^{\min} - V_{\Pi}^{\min}\right)1n\left(s - V_{\Pi}^{\max}\right) + \frac{1}{6}\left(V_{\Delta}^{\min} + V_{\Pi}^{\min}\right) + A, \qquad \text{(III.lla)}$$

$$D_{\Delta}(s) = \frac{1}{3}s - \frac{1}{6}\left(V_{\Delta}^{\min} - V_{\Pi}^{\min}\right)1n\left(s - V_{\Pi}^{\max}\right) + \frac{1}{6}\frac{\left(V_{\Delta}^{\min} - V_{\Pi}^{\min}\right)^2}{s - V_{\Pi}^{\min}} - A, \qquad \text{(III.11b)}$$

where A is the constant of integration. These particular results hold for the $V_{\Pi}^{\min} < V_{\Delta}^{\min}$ case of the uniform rectangular distribution. Similar results can readily be obtained for the cases $V_{\Pi}^{\min} = V_{\Delta}^{\min}$ and $V_{\Pi}^{\min} > V_{\Delta}^{\min}$.

d. A simple correlated distribution: continuous call rule

The purpose of the present exercise is to exhibit an example in which the valuations are correlated but, nevertheless, the damages curve may be explicitly obtained.

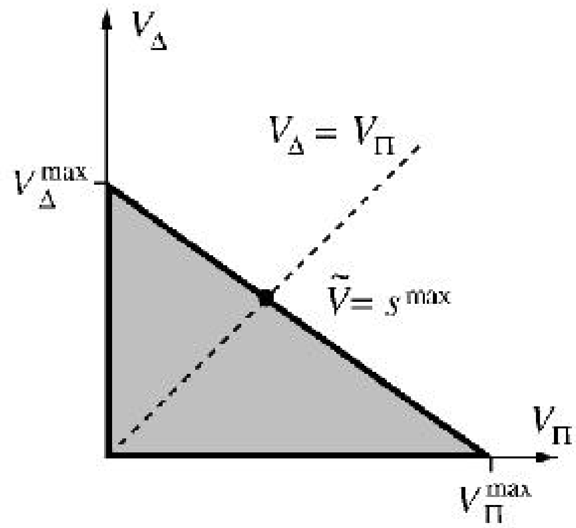

Fig. 7.— Uniform distribution in a triangle.

Here we take the joint probability distribution to be uniform in the equilateral triangle with corners $(0,0), (V_\Pi^{\max}, 0)$ and $(0, V_\Delta^{\max})$. Alternatively, we view this region as the first quadrant $(V_\Pi > 0, V_\Delta > 0)$ bounded by the line $(V_\Pi / V_\Pi^{\max}) + (V_\Delta / V_\Delta^{\max}) = 1$. The intersection with the line $V_\Delta = V_\Pi$ determines $s_{\max} = \tilde{V} \equiv V_\Pi^{\max} V_\Delta^{\max} \big/ (V_\Pi^{\max} + V_\Delta^{\max})$. For this distribution it is straightforward to show that $_{\Pi}(s) = \quad (s)\, V_{\Pi}^{\max} \big/ \left(V_{\Pi}^{\max} + V_{\Delta}^{\max}\right)$ and $_{\Delta}(s) = \quad (s)\, V_{\Delta}^{\max} \big/ \left(V_{\Pi}^{\max} + V_{\Delta}^{\max}\right)$, where

$(s) \equiv 1 \big/ \left(\widetilde{V} - s\right)$. Consequently, the equation of Table 4 for $\widetilde{D}(s)$ takes on the simple form

$$\widetilde{D}'(s) = \frac{\widetilde{D}(s) - s}{\widetilde{V} - s}, \qquad \text{(III.12)}$$

which can immediately be rewritten as

$$\frac{d}{ds}\left[(\tilde{V} - s)\tilde{D}(s)\right] = -s, \qquad \text{(III.13)}$$

and hence integrated to give, upon applying the boundary condition $\widetilde{D}\left(\widetilde{V}\right) = \widetilde{V}$,, the result

$$\tilde{D}(s) = (\tilde{V} + s)/2. \qquad \text{(III.14)}$$

Substituting this result into the differential equation for $D_{\Pi}(s)$ gives

$$D'_{\Pi}(s) = \frac{V_{\Pi}^{\max}}{V_{\Pi}^{\max} - V_{\Delta}^{\max}} \frac{(\tilde{V} + s)/2 - s}{\tilde{V} - s} = \frac{1}{2} \frac{V_{\Pi}^{\max}}{V_{\Pi}^{\max} - V_{\Delta}^{\max}}. \qquad \text{(III.15)}$$

Integrating and using $D_{\Delta}(s) = \widetilde{D}(s) - D_{\Pi}(s)$ then gives the damages functions

$$D_{\Pi}(s) = \frac{1}{2}\frac{V_{\Pi}^{\max}}{V_{\Pi}^{\max} - V_{\Delta}^{\max}}s + A, \qquad \text{(III.16a)}$$

$$D_{\Delta}(s) = \frac{1}{2}\frac{V_{\Delta}^{\max}}{V_{\Pi}^{\max} - V_{\Delta}^{\max}}s + \tilde{V} - A, \qquad \text{(III.16b)}$$

where A is, again, the constant of integration; these results can be expressed in the more symmetrical form

$$D_{\Pi}(s) = \frac{V_{\Pi}^{\max}}{2}\frac{V_{\Delta}^{\max} = s}{V_{\Pi}^{\max} = V_{\Delta}^{\max}} + A, \qquad \text{(III.17a)}$$

$$D_{\Delta}(s) = \frac{V_{\Delta}^{\max}}{2}\frac{V_{\Pi}^{\max} + s}{V_{\Pi}^{\max} + V_{\Delta}^{\max}} - A. \qquad \text{(III.17b)}$$

4. Uniform triangular distribution: continuous put rule

This example turns out to be the least interesting of the four we have chosen. As long as the lower left corner of the region in which the distribution is nonzero has the form of the rectangle, the expressions for $_{\Pi}(s)$ and $_{\Delta}(s)$ are the same as for the rectangle (with $V_{\Pi}^{\min} = V_{\Delta}^{\min} = 0$ in this example). Hence, the damages functions are identical to those found for the rectangle. To rephrase this more sharply, the damages functions do not depend on any probability weight that lies outside

the box $V_{\Pi} < s_{\max,}V_{\Delta} < s_{\max}$ (for the continuous put rule, or the box $V_{\Pi} > s_{\min}, V_{\Delta} > s_{\min}$ (for the continuous call rule).

For completeness of the solution we do need to specify the damages for $s > s^{\max}$. In fact the appropriate choice is to set $D_{\Pi}(s) = D_{\Pi}\left(s^{\max}\right), D_{\Delta}(s) = D_{\Delta}\left(s^{\max}\right)$ for $s > s^{\max}$.

5. General solution: continuous call rule

In this section we shall derive explicit formulas for the damages functions which can be used for arbitrary distributions of valuations, correlated or otherwise. As we did for the examples, we first solve for $\widetilde{D}(s)$, which can be seen from the equations in Table 4 to obey the differential equation

$$\widetilde{D}'(s) \quad \lambda(s)[\widetilde{D}(s) - s \qquad \text{(III.18)}$$

together with the boundary condition $\widetilde{D}\left(s^{\max}\right) = s_{\max}$. As the differential equation is a first-order ordinary one, we use the method of integrating factors, by which we find that

$$\frac{d}{ds}\left[\widetilde{D}(s) e^{\int_{s}^{s\max} \ (u)du}\right] = -s \ \ (s) e^{\int_{s}^{s\max} \ (u)du}, \qquad \text{(III.19)}$$

and hence that

$$\tilde{D}(s) e^{\int_s^{s\max} \quad (u)du} = K + \int_s^{s\max} dt\, t \quad (t) e^{\int_t^{s\max} \quad (u)du} \quad , \qquad \text{(III.20)}$$

where K is the constant of integration.

Next, we determine K by applying the boundary condition. To do this, we consider the limit $s \to s_{\max}$, bearing in mind that $\ (s)$ *is* singular in this limit. By making use of the elementary identity[43]

$$e^{\int_s^{s_{\max}} \lambda(u)du} = \int_s^{s_{\max}} dt\, \lambda(t)\, e^{\int_t^{s_{\max}} \lambda(u)du} , \qquad \text{(III.21)}$$

and observing that, in the limit $s \to s_{\max}$, we may replace *t* on the right hand side of the solution for $\tilde{D}(s)$ by $s_{\max}$, we recognize that the boundary condition is satisfied if $K = 0$. Thus, we arrive at the solution

$$\tilde{D}(s) = \int_s^{s_{\max}} dt\, t\, \lambda(t)\, e^{-\int_s^t \lambda(u)du} . \qquad \text{(III.22)}$$

To complete our task, we insert the formula for $D(s)$ into the differential equation obeyed by $D_\Pi(s)$ [we could equally well have used $D_\Delta(s)$] to obtain

$$D'_\Pi(s) = \lambda_\Pi(s) \int_s^{s_{\max}} dt\, (t - s)\lambda(t)\, e^{-\int_s^t \lambda(u)du} , \qquad \text{(III.23)}$$

[43]Obtained by observing the the r.h.s. is the integral of a total derivative.

where $_{\Pi}(s)$ is given in section A; integrating yields the result:

$$D_{\Pi}(s) = A + \int_{s_{\min}}^{s} dt\, \lambda_{\Pi}(t) \int_{t}^{s_{\max}} dt'\, (t' - t)\lambda(t')\, e^{-\int_{t}^{t'} \lambda(u)du}. \qquad \text{(III.24)}$$

As we now know $\widetilde{D}(s)$ and $D_{\Pi}(s)$, it is straightforward to construct $D_{\Delta}(s)$, using $D_{\Pi}(s) + D_{\Delta}(s) = \widetilde{D}(s)$. As mentioned at the start of this section, these damages formulas can be used for any joint probability distributions of valuations, the latter featuring through the quantities $_{\Pi}(s)$ and $_{\Delta}(s)$; see section A.

6. General solution: continuous put rule

We devote this section to deriving the general solution for the continuous put rule. As only slight modifications of the formalism for the continuous put rule are needed, the explanations will be brief. As we know from Table 4, the combined damages bb(s) satisfy

$$\widetilde{D}'(s) = \mu(s)[s - \widetilde{D}(s)], \qquad \text{(III.25)}$$

together with the boundary conditions by the method of integrating factors yields the following result:

$$\widetilde{D}(s) = \int_{s\,\min}^{s} dt \quad t \quad (t) e^{-\int_{t}^{s} \quad (u)du} \qquad \text{(III..26)}$$

Closer examination reveals that this form also satisfies the boundary condition at $s \rightarrow s_{\min}$. Inserting the result into the differential equation for $D_\Pi(s)$ from Table 2 and integrating, we obtain:

$$D_\Pi(s) = A + \int_{s\min}^{s} dt \quad {}_{\Delta}(t) \int_{s\min}^{t} dt'(t - t') \quad (t')e^{-\int_{t'}^{t} \, (u)du} \qquad \text{(III.27)}$$

The defendant's damages $D_\Delta(s)$ are obtained using $D_\Delta(s) = \widetilde{D}(s) - D_\Pi(s)$.

IV. Game theoretic formulation

In passing to the continuum limit of the iterated call and put rules, we have designed the damages functions in such a way that the plaintiff's and defendant's optimal strategies become $V_\Pi(s) = V_\Delta(s) = s$, i.e., each reveals his private information. The idea that incentive problems can be efficiently solved by designing a mechanism under which rational participants reveal their private information truthfully has become known as the *revelation principle* and was originally formulated by J. Mirrlees[44]. As we now discuss, one can use the revelation principle to construct a general formulation of the problem of efficient asset allocation in the context of liability rules.

We present a view of liability rules as *games* between the plaintiff and the defendant played according to rules stipulated by the court. Each player makes a *move* (for instance, announces a number that identifies one of his possible strategies), accounting for his private valuation and the common information at his disposal. The players move at the same time, and the court chooses the final asset holder

[44]James Mirrlees, *An Exploration in the Theory of Optimal Income Taxation*, REV. ECON. STUD. (1971), 15 J. Legal Stud. 93 (1986).

and the damages exchanged (e.g. by looking them up in a table having rows and columns corresponding to the plaintiff's and defendant's moves). In this strategic form, the players, who have full knowledge of the payoff table, solve the problem of finding their optimal strategies, and make their moves accordingly. However, they could equivalently delegate their decision-making to the court by revealing their private valuation, provided they are assured that the court will use reasoning identical to theirs and will make the corresponding moves on their behalf. It must be stressed that in making a decision on the plaintiff's behalf the court should pretend that it does not know the defendant's private valuation, and vice versa, in order to correctly mimic the litigants' behavior. The next step consists of combining the two steps—finding the optimal moves and determining the winner (who will become the owner of the asset) and the damages—into one. The court asks the litigants to submit their private information and uses these valuations to determine who is to be the asset holder and the damages. The added requirement is that the mechanism be *incentive compatible*[45] —the litigants must not be able to gain any advantage by misrepresenting their private information.

With this scheme in mind, if we are aiming at achieving the perfect efficiency, the court should allocate the asset to the party with the higher private valuation. If the announced valuations are s_Π and s_Δ, then the asset should go to the plaintiff if $s_\Pi > s_\Delta$ and to the defendant if $s_\Pi < s_\Delta$ (or to either party if $s_\Pi = s_\Delta$). Furthermore, the court sets the damages at $D(s_\Pi, s_\Delta)$; the function D has to be crafted in an incentive-compatible way. Note that the function $D(s_\Pi, s_\Delta)$ will, in general, be discontinuous across

[45] *See, e.g.*, *infra* note 36.

$s_\Pi = s_\Delta$. It will, therefore, be convenient to work with two functions, $D_\Pi\left(s_\Pi, s_\Delta\right)$ and $D_\Delta\left(s_\Pi, s_\Delta\right)$, defined only for $s_\Pi \geq s_\Delta$ and $s_\Pi \leq s_\Delta$, respectively.

To find the necessary restrictions placed on $D_\Pi\left(s_\Pi, s_\Delta\right)$ and $D_\Delta\left(s_\Pi, s_\Delta\right)$, let us assume that the players have the private valuations V_Π and V_Δ, and that their optimal strategies $s_\Pi\left(V_\Pi\right)$ and $s_\Delta\left(V_\Delta\right)$ are not necessarily revealing. (They are revealing if $s_\Pi\left(V_\Pi\right) = V_\Pi$ and $s_\Delta\left(V_\Delta\right) = V_\Delta$.) We shall then enforce the condition that if any one's strategy is revealing, the opponent's best response is to follow a revealing strategy. In other words, we shall seek damages functions such that this scenario is always realized.

As usual, we choose $\int\left(V_\Pi, V_\Delta\right)$ to denote the joint probability distribution governing the valuations. For convenience, we use the Heaviside function $\theta(x)$ [46] as a tool for restricting the regions of integration. For instance, we would multiply the integrand by $\theta\left(V_\Pi - V_\Delta\right)$ to restrict the region of integration to the half-plane $V_\Pi > V_\Delta$ We also make use of θ 's formal derivative (known as the Dirac - function) $\theta'(x) \equiv \quad (x)$. The latter has a meaning only inside an integral, so that $\int dx \quad (x - a) f(x) = f(a)$. We regard the litigants' expected payoffs, $_\Pi$ and $_\Pi$, as entities

[46] The Heaviside function may be defined via $\theta(x) \equiv \begin{Bmatrix} 1 & x>0 \\ 0 & x\leq 0 \end{Bmatrix}$.

that depend on the strategy functions $s_\Pi\left(V_\Pi\right)$ and $s_\Delta\left(V_\Delta\right)$ [47]. In the present context, in which the asset goes to the higher bidder, the expected payoff functionals are given by

$$\begin{aligned} \pi_\Pi[s_\Pi(\cdot), s_\Delta(\cdot)] &= \int\!\!\int dV_\Pi\, dV_\Delta\, f(V_\Pi, V_\Delta)\, \{\theta(s_\Pi - s_\Delta)[V_\Pi - D_\Pi(s_\Pi, s_\Delta)] \\ &\qquad + \theta(s_\Delta - s_\Pi) D_\Delta(s_\Pi, s_\Delta)\}, \qquad \text{(IV.1)} \\ \pi_\Delta[s_\Pi(\cdot), s_\Delta(\cdot)] &= \int\!\!\int dV_\Pi\, dV_\Delta\, f(V_\Pi, V_\Delta)\, \{\theta(s_\Pi - s_\Delta) D_\Pi(s_\Pi, s_\Delta) \\ &\qquad + \theta(s_\Delta - s_\Pi)[V_\Delta - D_\Delta(s_\Pi, s_\Delta)]\}. \qquad \text{(IV.2)} \end{aligned}$$

The restriction on the damages, which we are seeking, is chosen so that the revealing strategies $s_\Pi\left(V_\Pi\right) = V_\Pi$ and $s_\Delta\left(V_\Delta\right) = V_\Delta$ make the litigants' expected payoff stationary with respect to variations in strategy:

$$\frac{\delta}{\delta s_\Pi(V_\Pi)} \pi_\Pi[s_\Pi(\cdot), s_\Delta(\cdot)] = \frac{\delta}{\delta s_\Delta(V_\Delta)} \pi_\Delta[s_\Pi(\cdot), s_\Delta(\cdot)] = 0. \qquad \text{(IV.3)}$$

Note that we are using *functional derivatives*,[48] rather than conventional ones. Functional derivatives can be thought of as partial derivatives in a multidimensional space of infinitely many variables [i.e.

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[47]It is customary to refer to entities that depend on functions, rather than, say, variables, as *functionals* of those functions.

[48]*See*, e.g., C. Nash, *Relativistic Quantum Fields* (Academic Press, London, 1978).

$s_1 = s(V_1), s_2 = s(V_2), \ldots$, where $V_1, V_2, \ldots$ enumerate all real values]. In the simple case in which $[s(V)]$ is expressible in the form

$$\pi[s(\cdot)] = \int dV \, p(s(V)), \qquad \text{(IV.4)}$$

the functional derivative turns out to be computable via[49] :

$$\frac{[s(\cdot)]}{s(V)} = \frac{dp}{ds}\bigg|_{s=s(V)} \qquad \text{(IV.5)}$$

The stationarity conditions Eq. (IV.3) are, indeed, expressible in this form.

Intuitively, the condition $_{\Pi} \geq / \; s_{\Pi}(V_{\Pi}) = 0$ is solved by first fixing V_{Π} and then optimizing the payoff by adjusting s_{Π}. The procedure, repeated for all V_{Π}, yields the function equation

$$0 = \int dV_{\Delta} f(V_{\Pi}, V_{\Delta}) \{ \Theta(s_{\Pi} - s_{\Delta}) D_{\Pi}^{(\Pi)}(s_{\Pi}, s_{\Delta}) + \Theta(s_{\Delta} - s_{\Pi}) D_{\Delta}^{(\Pi)}(s_{\Pi}, s_{\Delta})$$

$$+ \; (s_{\Pi} - s_{\Delta}) [V_{\Delta} - D_{\Pi}(s_{\Pi}, s_{\Delta}) - D_{\Delta}(s_{\Pi}, s_{\Delta})] \}, \qquad \text{(IV.6a)}$$

where $D^{(\Pi)}(s_{\Pi}, s_{\Delta})$ is used to denote the partial derivative $D(s_{\Pi}, s_{\Delta}) / \; s_{\Pi}$. Similarly, optimizing the defendant's payoff we obtain

[49] Note that the functional derivative of a scalar is itself a function (of *V*).

$$0 = \int dV_{\Pi} f\left(V_{\Pi}, V_{\Delta}\right)\{\Theta\left(s_{\Pi} - s_{\Delta}\right)D_{\Pi}^{(\Delta)}\left(s_{\Pi}, s_{\Delta}\right) - \Theta\left(s_{\Delta} - s_{\Pi}\right)D_{\Delta}^{(\Delta)}\left(s_{\Pi}, s_{\Delta}\right)$$

$$+ \left(s_{\Pi} - s_{\Delta}\right)\left[V_{\Delta} - D_{\Pi}\left(s_{\Pi}, s_{\Delta}\right) - D_{\Delta}\left(s_{\Pi}, s_{\Delta}\right)\ \right]\} \qquad \text{(IV.6b)}$$

where $D^{(\Delta)}\left(s_{\Pi}, s_{\Delta}\right)$ denotes the partial derivative $D\left(s_{\Pi}, s_{\Delta}\right)/\ s_{\Delta}$. The next step is to determine the constraints on D_{Π} and D_{Δ} imposed by the demand that revealing strategies are an equilibrium. Thus, we insert $s_{\Pi}\left(V_{\Pi}\right) = V_{\Pi}$ and $s_{\Delta}\left(V_{\Delta}\right) = V_{\Delta}$ into Eqs. (IV.6a) and (IV.6b), arriving at the conditions

$$0 = \int dV_{\Delta}\, f(V_{\Pi}, V_{\Delta})\left\{ -\theta(V_{\Pi} - V_{\Delta})D_{\Pi}^{(\Pi)}(V_{\Pi}, V_{\Delta}) + \theta(V_{\Delta} - V_{\Pi})D_{\Delta}^{(\Pi)}(V_{\Pi}, V_{\Delta})\right\}$$
$$-f(V_{\Pi}, V_{\Pi})\left\{V_{\Pi} - D_{\Pi}(V_{\Pi}, V_{\Pi}) - D_{\Delta}(V_{\Pi}, V_{\Pi})\right\}, \qquad \text{(IV.7a)}$$
$$0 = \int dV_{\Pi}\, f(V_{\Pi}, V_{\Delta})\left\{\theta(V_{\Pi} \quad V_{\Delta})D_{\Pi}^{(\Delta)}(V_{\Pi}, V_{\Delta}) \quad \theta(V_{\Delta} \quad V_{\Pi})D_{\Delta}^{(\Delta)}(V_{\Pi}, V_{\Delta})\right\}$$
$$f(V_{\Delta}, V_{\Delta})\left\{V_{\Delta} \quad D_{\Pi}(V_{\Delta}, V_{\Delta}) \quad D_{\Delta}(V_{\Delta}, V_{\Delta})\right\}. \qquad \text{(IV.7b)}$$

Recall that section III was devoted to determining the conditions obeyed by the damages functions for the continuous call and continuous put rules. We can recover these conditions from the more general structure, just developed. To see this, recall that, under the continuous call rule, the damages were determined by the losing bid. In the present language, this reads

$$D_{\Pi}(V_{\Pi}, V_{\Delta}) - D_{\Pi}(V_{\Delta}), \qquad \text{(IV.8a)}$$

$$D_{\Delta}(V_{\Pi}, V_{\Delta}) = D_{\Delta}(V_{\Pi}). \qquad \text{(IV.8b)}$$

Hence, not only do $D_{\Pi}^{(\Pi)} = D_{\Delta}^{(\Delta)} = 0$, but also $D_{\Pi}^{(\Delta)}$ and $D_{\Delta}^{(\Pi)}$ can be taken out of the integrations in Eqs. (IV.7a) and (IV.7b), which then become

$$0 = D'_{\Delta}(V_{\Pi}) \int dV_{\Delta}\, f(V_{\Pi}, V_{\Delta})\theta(V_{\Delta} - V_{\Pi}) - f(V_{\Pi}, V_{\Pi})[V_{\Pi} - D_{\Pi}(V_{\Pi}) - D_{\Delta}(V_{\Pi})], \tag{IV.9a}$$

$$0 = D'_{\Pi}(V_{\Delta}) \int dV_{\Pi}\, f(V_{\Pi}, V_{\Delta})\theta(V_{\Pi} - V_{\Delta}) + f(V_{\Delta}, V_{\Delta})[V_{\Delta} - D_{\Pi}(V_{\Delta}) - D_{\Delta}(V_{\Delta})], \tag{IV.9b}$$

i.e., precisely the conditions on $D_{\Pi}(V_{\Delta})$ and $D_{\Delta}(V_{\Pi})$ obtained in section III. Similarly, for the case of the continuous put rule, by setting $D_{\Pi}(V_{\Pi}, V_{\Delta}) = D_{\Pi}(V_{\Pi})$ and $D_{\Delta}(V_{\Pi}, V_{\Delta}) = D_{\Delta}(V_{\Delta})$ we recover precisely the conditions on $D_{\Pi}(V_{\Pi})$ and $D_{\Delta}(V_{\Delta})$ obtained in section III. To conclude this section, let us make a few remarks about the validity of these results. We have proven the stationarity of the revealing strategies but have not demonstrated that they constitute maxima of the expected payoffs. Thus, the conditions we have found are necessary, but not (necessarily) sufficient for the existence of the revealing Nash equilibrium. In B, we shall prove that these conditions are also sufficient for the uncorrelated case: $f(V_{\Pi}, V_{\Delta}) = f_{\Pi}(V_{\Pi}) f_{\Delta}(V_{\Delta})$. The uniqueness of the Nash equilibrium remains an open problem even for the uncorrelated case.

V. Conclusion

In this Paper we have proposed two new types of liability rule ~the continuous call and the continuous put. In contrast to traditional liability rules, which depend on a single damage parameter *D,* the continuous rules are specified by the functions $D_{\Pi}(s)$ and $D_{\Delta}(s)$ These rules can be thought of as the infinite-stage limit of the iterated call and iterated put rule or, equivalently, as an entirely new procedure, akin to the Vickrey and first-best auctions. We have shown that these rules are rich, in the sense that (a) all rules encountered, to date, can be expressed as special cases of either the continuous call or the continuous put rule, and (b) by appropriately choosing $D_{\Pi}(s)$ and $D_{\Delta}(s)$, it is possible to design a mechanism capable of achieving the first-best efficiency, assuming that the plaintiff and the defendant are both rational players. We have also reformulated the problem using the revelation principle, and have obtained the conditions satisfied by the damages function $D(V_{\Pi}, V_{\Delta})$, we have also shown how the continuous call and put rules manifest themselves as two special cases of this most general mechanism.

The striking similarities between our results for the continuous call and continuous put rules can be explained as a manifestation of the *duality* of these two mechanisms. Indeed, if we assume the existence of an upper bound on private valuations, i.e., the price *M* such that, with probability 1, both $V_{\Pi}, V_{\Delta} \leq M$ [That $V_{\Pi}, V_{\Delta} \leq 0$ is implicit.] To each asset, we may assign the corresponding liability a contract to acquire the asset for the amount *M.* For bilateral disputes, the right to be free from such liability represents the asset dual to the original one.[50] It can be argued that the outcome of the continuous call rule is equivalent to the outcome of the continuous put for dual assets, and the outcome of the continuous put rule for the original assets is the same at that of the continuous call for the dual assets. Thus, one might say that the distinction between the two is only superficial. It remains to be seen whether any of the more general damages functions that satisfy the efficiency conditions of section IV correspond to rules of any practical value.

A famous (negative) result, due to Myerson and Satterthwaite[51] states that it is impossible to have an efficient trading mechanism. To be precise, whenever domains of non-zero probability overlap, no mechanism is possible that is both (a) individually rational [for players to participate in the procedure] and (b) first-best efficient. Work by Chatterjee and Samuelson[52] relaxes the second constraint in order to solve the problem for the case of a distribution that is uniform in the unit [0; 1] x [0; 1] square. The present Paper relaxes the first constraint.[53] While courts can assure that the disputants' expected payoffs (given the court's knowledge of the general value probability distributions) are positive, the payoffs of privately informed playoffs can be negative. But this potential participation problem is equally acute for traditional

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[50]This feature is reminiscent of the call-put parity result for vanilla liability rule. See *supra* note 7 and *supra* note 10.

[51]R.B. Myerson and M.A. Satterhwaite, *Efficient Mechanisms for Bilateral Trading*, J. Of Econ. Theory 29, 265 (1983).

[52]K. Chatterjee and W. Samuelson, *Bargaining under Incomplete Information*, 31 Oper. Research 835, 837-38 (1983).

[53]Peter Cramton, Robert Gibbons, and Paul Klemperer, in Dissolving a Partnership Efficiently [Econometrica 55, 615 (1987)], formulate a set of conditions under which a partnership (i.e. three or more players dividing two or more assets) can efficiently be dissolved. They restrict their attention to symmetric, uncorrelated distributions, and formulate additional constraints on the distributions that make efficient allocation possible.

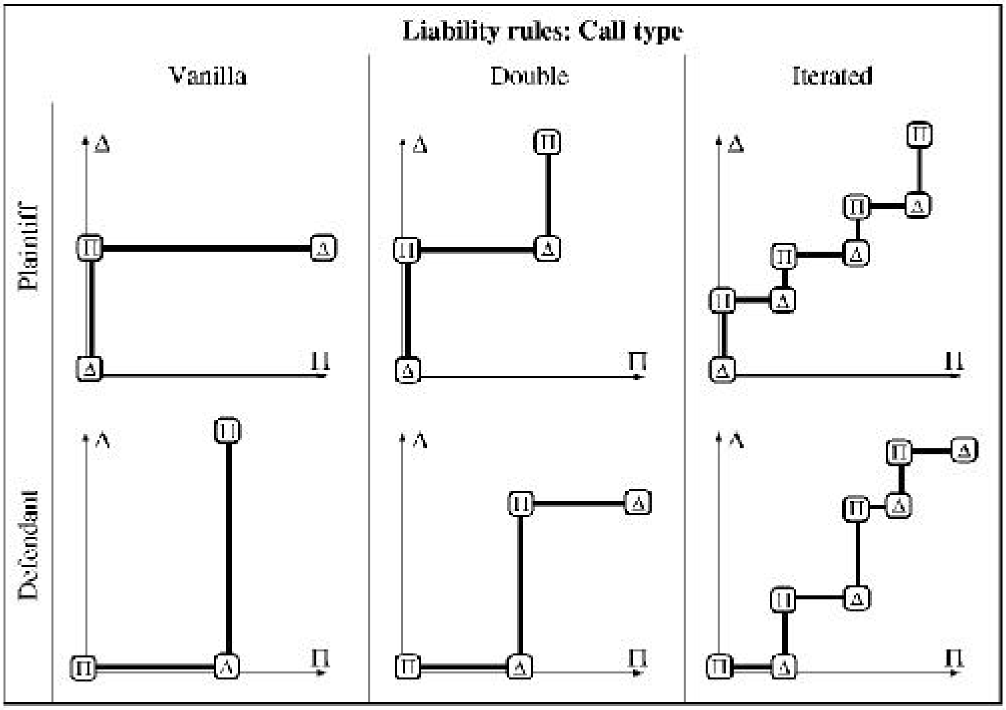
liability rules.[54]

Thus, in future work, it would be useful to analyze the extent to which a participation constraint limits the analysis. For now, it is at least clear that litigation often extinguishes the participation choice of one disputant. Defendants, by polluting (or by seeking a declaratory), can often force plaintiff participation in the mechanism. And plaintiffs, by suing, might often force defendant participation. The possible need to assure the participation of only one relevant disputant might mean that it may be appropriate for courts to divide the total payoffs (through its setting of A) in a way that assures that the appropriate range of disputes is subjected to the continuous call or continuous put mechanism.

Appendix: Demonstration of Nash equilibrium for the uncorrelated case

The purpose of this appendix is to show that, provided that the distribution of valuations is uncorrelated, and provided that the damages functions obey the conditions given in Eqs. (IV.7a) and (IV.7b), the strategy in which both players reveal their private information is a Nash equilibrium. To see this, consider the equations obeyed by the damages functions for the uncorrelated case:

[54] Saul Levmore, *Unifying Remedies: Property Rules, Liability Rules, and Startling Rules*, 106 Yale L.J. 2149 (1997).


Liability rules: Call type
Vanilla
Double
Iterated
Plaintiff
Defendant
Δ
Π

$$0 = \int dV_{\Delta}\, f_{\Delta}(V_{\Delta}) \left\{ \theta(V_{\Pi} - V_{\Delta})\, D_{\Pi}^{(\Pi)}(V_{\Pi}, V_{\Delta}) + \theta(V_{\Delta} - V_{\Pi})\, D_{\Delta}^{(\Pi)}(V_{\Pi}, V_{\Delta}) \right\}$$
$$+ f_{\Delta}(V_{\Pi})\, [V_{\Pi} - D_{\Pi}(V_{\Pi}, V_{\Pi}) - D_{\Delta}(V_{\Pi}, V_{\Pi})], \qquad \text{(B1)}$$
$$0 = \int dV_{\Pi}\, f_{\Pi}(V_{\Pi}) \left\{ \theta(V_{\Pi} - V_{\Delta})\, D_{\Pi}^{(\Delta)}(V_{\Pi}, V_{\Delta}) - \theta(V_{\Delta} - V_{\Pi})\, D_{\Delta}^{(\Delta)}(V_{\Pi}, V_{\Delta}) \right\}$$
$$+ f_{\Pi}(V_{\Delta})\, [V_{\Delta} - D_{\Pi}(V_{\Delta}, V_{\Delta}) - D_{\Delta}(V_{\Delta}, V_{\Delta})]. \qquad \text{(B2)}$$

Assume that the plaintiff's valuation is V_{Π}. The plaintiff's payoff from making the bid s_{Π} can, provided that the defendant uses the revealing strategy $s_{\Delta} = V_{\Delta}$, be expressed as

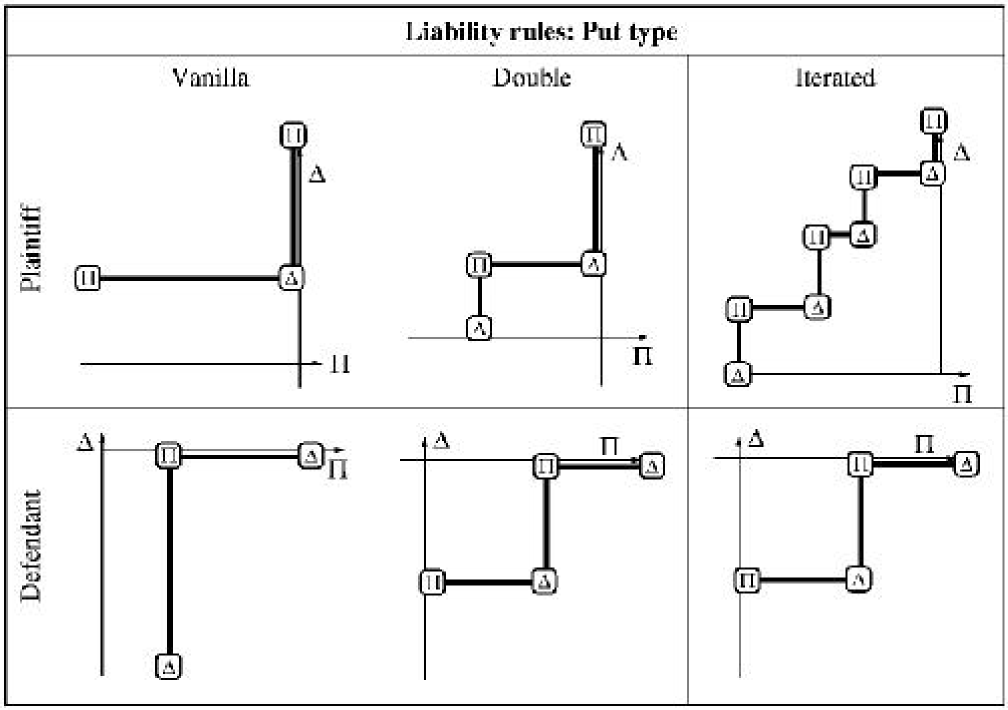
$$\pi_{\Pi}[s_{\Pi}|V_{\Pi}] = \int dV_{\Delta}\, f_{\Delta}(V_{\Delta}) \{ \theta(s_{\Pi} - V_{\Delta}) [V_{\Pi} - D_{\Pi}(s_{\Pi}, V_{\Delta})] - \theta(V_{\Delta} - s_{\Pi})\, D_{\Delta}(s_{\Pi}, V_{\Delta}) \}. \qquad \text{(B3)}$$

Now, we can safely assume that s_{Π} is such that $f_{\Pi}(s_{\Pi}) > 0$.[55] Then, by differentiating the plaintiff's payoff ${}_{\Pi}[s_{\Pi}|V_{\Pi}]$ with respect to s_{Π}, and comparing the result with Eq. (IV.7a) for damages (having substituted $v_{\Pi} = s_{\Pi}$ in the latter), we can see that

$$\frac{d}{ds_{\Pi}} \pi_{\Pi}[s_{\Pi}|V_{\Pi}] = f_{\Delta}(s_{\Pi})\, (V_{\Pi} - s_{\Pi}), \qquad \text{(B4)}$$

[55] For instance, the court can assess a huge fine [setting $D_{\Pi}(s_{\Pi}, v_{\Delta}) = +\infty$ or $D_{\Delta}(s_{\Pi} v_{\Delta}) = -\infty$] on a player (plaintiff) who reports an impossible private valuation.

i.e., $d \;_{\Pi} / ds_{\Pi} > 0$ when $s_{\Pi} < V_{\Pi}$ and $d \;_{\Pi} / ds_{\Pi} < 0$ when $s_{\Pi} > V_{\Pi}$. This proves that $s_{\Pi} = V_{\Pi}$ is the plaintiff's best response . This analysis, applied to the defendant's payoff, shows that $s_{\Delta} = V_{\Delta}$ is the defendant's best response, provided the plaintiff uses the revealing strategy. Therefore, the revealing strategies do, indeed, form a Nash equilibrium.


Liability rules: Put type
Vanilla
Double
Iterated
Plaintiff
Defendant